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Cutting Planes for Multilinear Optimization

Matthias Walter (Uni Twente)
Joint work with Emily Schutte (Uni Luxembourg)
and Alberto del Pia (Uni Wisconsin-Madison)

Funding support from the Dutch Research Council (NWO) on grant number OCENW.M20.151.

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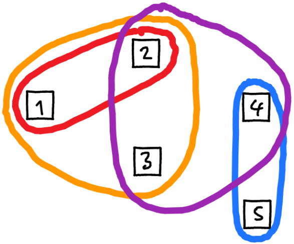
Definition – Multilinear polytope [Del Pia, Khajavirad '16; Buchheim, Crama, Rodríguez-Heck '16]

Let $G = (V, E)$ be a hypergraph. The **multilinear polytope** of G is the polytope

$$\text{ML}(G) := \text{conv}\left\{(x, y) \in \{0,1\}^V \times \{0,1\}^E : y_e = \prod_{v \in e} x_v \text{ for each hyperedge } \{u, v\} \in E\right\}.$$

Example:



$y_{\{1,2\}} = x_1 \cdot x_2$

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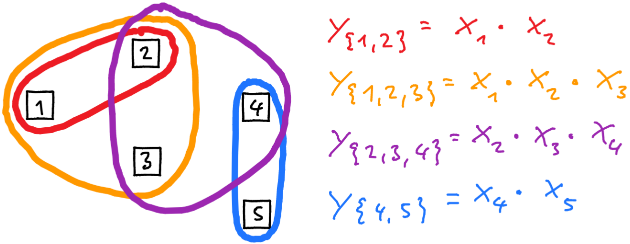

Remarks:

- For binary $x \in \{0,1\}^V$, the original optimization problem (without auxiliary variables) is called pseudo-boolean optimization problem. [Hammer, Hansen, Simeone '84; Boros, Hammer '91]
- For each hyperedge $e = \{v_1, v_2, \ldots, v_k\}$, we have the logic AND constraint $y_e = x_{v_1} \wedge x_{v_2} \wedge \cdots \wedge x_{v_k}$.

Special Case: Boolean Quadric Polytopes

Definition – Boolean quadric polytope **[Padberg '88]**

Let $G = (V, E)$ be a graph. The **boolean quadric polytope** of G is the polytope

$$\mathrm{BQP}(G) := \mathrm{conv}\left\{(x,y) \in \{0,1\}^V \times \{0,1\}^E : y_{\{u,v\}} = x_u \cdot x_v \text{ for each edge } \{u,v\} \in E\right\}.$$

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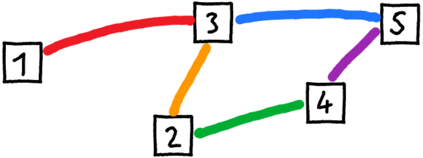

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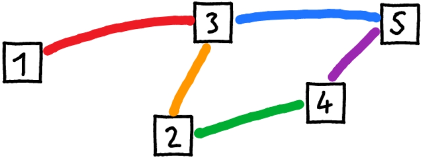

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Remarks:

- Equivalent to CUT polytope of related graph. [Barahona, Mahjoub '86; De Simone '90]
- Can be used to minimize a quadratic function $q(x)$ over $x \in \{0,1\}^n$, also known as "quadratic unconstrained binary optimization" (QUBO).
- Optimization over BQP is NP-hard in general.

Standard Relaxation

Proposition – Standard relaxation [Fortet '60; Glover, Woolsey '74]

Let $G = (V, E)$ be a hypergraph. The polytope $\text{SR}(G)$ defined by

$$0 \leq y_e \leq x_v \leq 1 \qquad \forall v \in e \in E \qquad (1a)$$

$$y_e + \sum_{v \in e} (1 - x_v) \geq 1 \qquad \forall e \in E \qquad (1b)$$

yields an IP formulation, i.e., $\text{SR}(G) \cap \mathbb{Z}^{V+E} = \text{ML}(G) \cap \mathbb{Z}^{V+E}$.

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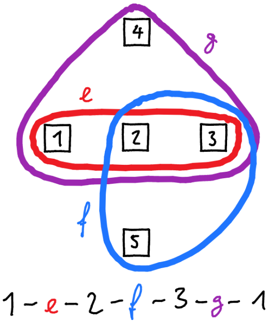
Berge cycle:

$v_1, e_1, v_2, e_2, \ldots, v_k, e_k, v_1$ with:

- $v_i \in V$ are distinct nodes.
- $e_i \in E$ are distinct edges.
- $v_i \in e_{i-1} \cap e_i$ for each i



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Theorem – Perfect formulation [Del Pia, Khajavirad '16; Buchheim, Crama, Rodríguez-Heck '16]

$\mathrm{SR}(G) = \mathrm{ML}(G)$ holds if and only if G is **Berge-acyclic**.

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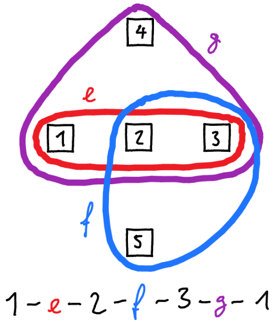

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Computational experience:

- Bounds obtained from $\text{SR}(G)$ are often very weak.
  [Luedtke, Namazifar, Linderoth '12; Bao, Khajavirad, Sahinidis, Tawarmalani '14]

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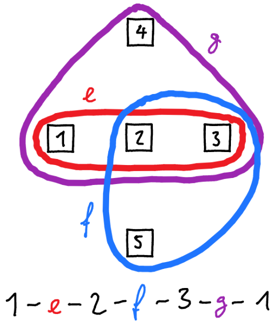
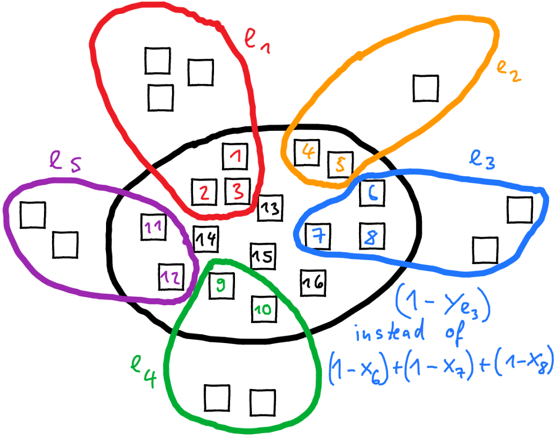


Flower Relaxation



Definition – Flower relaxation [Del Pia, Khajavirad '18]

The k-**flower inequality centered at** f **with neighbors** $e_1, e_2, \ldots, e_k$ is

$$y_f + \sum_{i=1}^{k}(1 - y_{e_i}) + \sum_{v \in R}(1 - x_v) \geq 1, \qquad (2)$$

where $R := f \setminus \bigcup_{i=1}^{k} e_i$ contains all nodes of v that are not in a leaf. We denote by FR(G) the standard relaxation SR(G), augmented by all 1-flower and all 2-flower inequalities.

For comparison: $y_f + \sum_{v \in f}(1 - x_v) \geq 1 \qquad (1b)$

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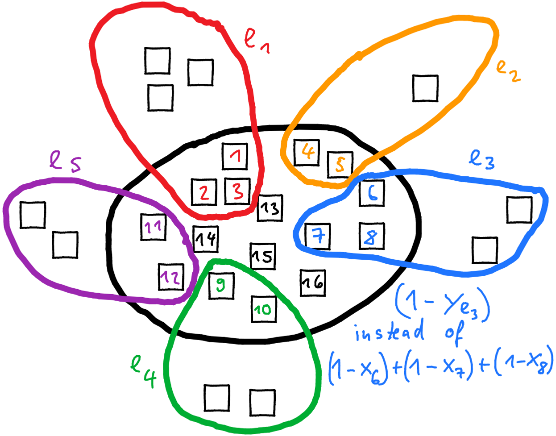

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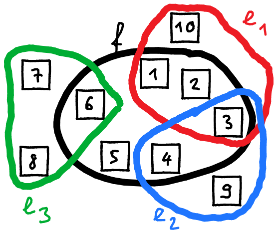
Special case:

- 1-flower inequalities were independently introduced as **2-link inequalities**. [Crama, Rodríguez-Heck '17]

Flower vs. Recursive McCormick Relaxations

Extended flower inequalities:

- Inequality is valid even if leaves overlap!



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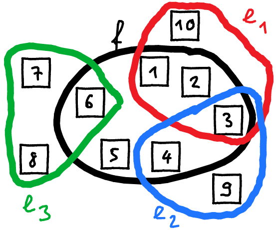

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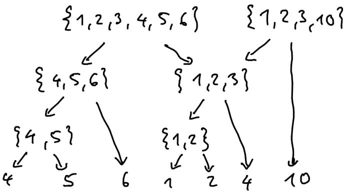
Recursive McCormick formulations:

- Add auxiliary variables y_I for intermediate sets I.



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Linearization:

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$$y_{\{1,2,3,4,5,6\}} = y_{\{4,5,6\}} \cdot y_{\{1,2,3\}}$$
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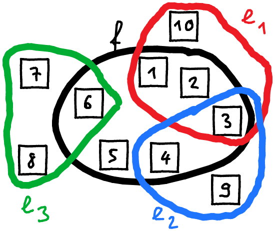

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Theorem – Flower power [Khajavirad '23]

The **extended flower relaxation** is at least as strong as any **recursive McCormick relaxation**.

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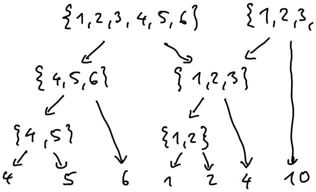

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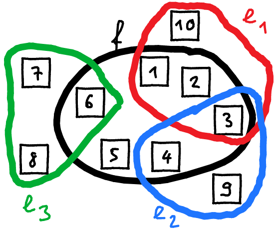

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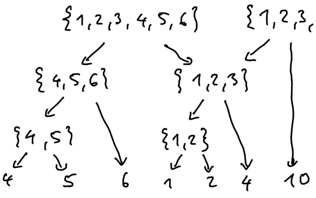

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Theorem – McCormick strikes back [Schutte, Walter '24]

The **extended flower relaxation** is equal to the intersection of all **recursive McCormick relaxations**.

Faster Separation of Flower Inequalities

A useful parameter:

- By $r \in \mathbb{N}$ we denote the **rank** of G, defined as the maximum size of an edge.

Theorem – Separation via Matching **[Del Pia, Khajavirad '18]**

For γ-acyclic hypergraphs, the separation problem can be solved in time $\mathcal{O}(r|E|^2(|V|+|E|))$.

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Proof idea:

- For each center edge f, enumerate sets of neighbors.

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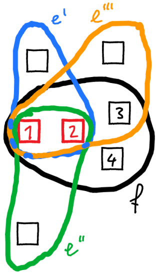

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Theorem – New algorithm **[Del Pia, Walter '24+]**

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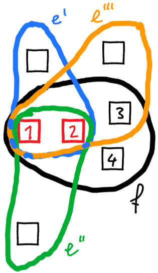

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- Per **overlap set** $U \coloneqq e \cap f$, store $\min\{(1 - y_e) \mid e \supseteq U\}$.
- For each center edge f, enumerate contained overlap sets. Minimizing edges are the neighbors.

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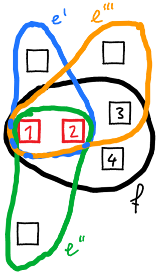

Definition – Odd β-cycle inequalities **[Del Pia, Di Gregorio '19]**

Definition 2. Consider a hypergraph $G = (V, E)$, let $C = v_1, e_1, v_2, \dots, v_m, e_m, v_1$ be a β-cycle in G, and let E^-, E^+ be a partition of $E(C)$ such that $k := |E^-|$ is odd and $e_1 \in E^-$. Let $D := \{e_{p+1}, e_{p+2}, \dots, e_m\}$, where e_p is the last edge in C that belongs to E^-. We denote by $f_1, \dots, f_k$ the subsequence of $e_1, \dots, e_m$ of the edges in E^-. Let $S_1 := (\cup_{e \in E^-} e) \setminus \bigcup_{e \in E^+} e$ and $S_2 := V(C) \setminus \bigcup_{e \in E^-} e$. With this notation in place, we make the following assumptions:

(a) Every node $v \in \bigcup_{i=1}^m e_i$ is contained in at most two edges among $e_1, \dots, e_m$.
(b) For every edge $e_i \in E^+ \setminus D$, every edge in E^- adjacent to e_i (if any) is either e_{i-1} or e_{i+1}.
(c) No edge in D is adjacent to an edge f_i with i even.
(d) At least one of the following two conditions holds:
(d1) For every $v \in S_1$, either v is contained in just one edge $e \in E^-$, or it is contained in two edges f_i, f_j with i odd and j even.
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We then define the odd β-cycle inequality corresponding to C and E^- as

$$\sum_{v \in S_1} z_v - \sum_{e \in E^-} z_e - \sum_{v \in S_2} z_v + \sum_{e \in E^+} z_e \leq |S_1| - |\{i \in \{1, \dots, m\} : e_i, e_{i+1} \in E^-\}| + \left\lfloor \frac{k}{2} \right\rfloor. \tag{2}$$

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Odd β-Cycle Inequalities

Definition – Odd β-cycle inequalities [Del Pia, Di Gregorio '19]

Definition 2. Consider a hypergraph $G = (V, E)$, let $C = v_1, e_1, v_2, \ldots, v_m, e_m, v_1$ be a β-cycle in G, and let E^-, E^+ be a partition of $E(C)$ such that $k := |E^-|$ is odd and $e_1 \in E^-$. Let $D := \{e_{p+1}, e_{p+2}, \ldots, e_m\}$, where e_p is the last edge in C that belongs to E^-. We denote by $f_1, \ldots, f_k$ the subsequence of $e_1, \ldots, e_m$ of the edges in E^-. Let $S_1 := (\cup_{e \in E^-} e) \setminus \bigcup_{e \in E^+} e$ and $S_2 := V(C) \setminus \bigcup_{e \in E^-} e$. With this notation in place, we make the following assumptions:

(a) Every node $v \in \bigcup_{i=1}^m e_i$ is contained in at most two edges among $e_1, \ldots, e_m$.

(b) For every edge $e_i \in E^+ \setminus D$, every edge in E^- adjacent to e_i (if any) is either e_{i-1} or e_{i+1}.

(c) No edge in D is adjacent to an edge f_i with i even.

(d) At least one of the following two conditions holds:

(d1) For every $v \in S_1$, either v is contained in just one edge $e \in E^-$, or it is contained in two edges f_i, f_j with i odd and j even.

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Let $G = (V, E)$ be a hypergraph. If there is a β-cycle C with a certain edge bipartition and some extra definitions satisfying some extra properties, then

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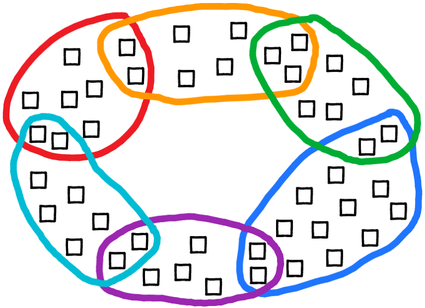
Theorem – Perfection [Del Pia, Di Gregorio '19]

For **cyclic hypergraphs** G, $\text{ML}(G)$ is completely described by $\text{FR}(G)$ plus odd β-cycle inequalities.

Cyclic hypergraph:



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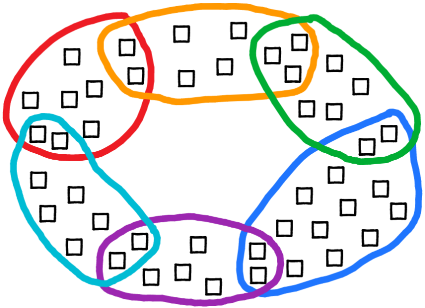


Simple / Relaxed Odd β-Cycle Inequalities

Definitions – Simple / Relaxed Odd β-cycle inequalities **[Del Pia, Walter '22/24+]**

Also skipped in this talk!

Remark:

- Both new definitions yields weaker inequalities in general!

Theorem – CG rank **[Del Pia, Walter '22/24+]**

Simple / relaxed odd β-cycle inequalities have Chvátal rank 2 (with respect to the standard relaxation SR).

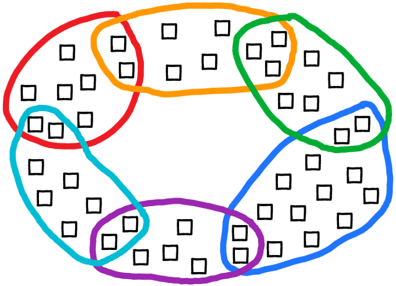
Theorem – Perfection **[Del Pia, Walter '22/24+]**

For **cyclic hypergraphs** G, $\text{ML}(G)$ is completely described by $\text{FR}(G)$ plus **simple / relaxed** odd β-cycle inequalities.

Theorem – Separation **[Del Pia, Walter '22/24+]**

For **arbitrary hypergraphs**, separation of **simple / relaxed** odd β-cycle inequalities can done in polynomial time.

Cyclic hypergraph:



Simple vs. Relaxed Odd β-Cycle Inequalities

Common idea:

- Patch together 2-flower, 1-flower and standard inequalities in cyclic fashion.
- Some coefficients in overlap cancel out, some add up.
- Scale by $1/2$ and round up the resulting inequality à la Chvátal.

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- Need to consider all edge pairs as nodes in auxiliary graph and run minimum-weight odd cycle algorithm.
- Running time: $\mathcal{O}(|E|^5 + |V|^2|E|)$

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Separation algorithm for relaxed:

- Need to consider overlap sets as nodes in auxiliary digraph and run minimum-weight odd cycle algorithm.
- Running time for fixed rank: $\mathcal{O}(|E|^2 \log |E|)$

Computational Results – Implementation

Implementation:

- SCIP plugin `sepa_multilinear`, inspecting all AND constraints:
- 1- and 2-flowers will be in SCIP 10.
- Simple/relaxed odd β-cycles are not mature, yet.

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- Time limit: 3600 s
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First insights:

- Fast 1- and 2-flower separation is orders of magnitude faster, so results for slow are skipped.
- Number of overlaps typically lies in ballpark of $|E|$.
- Odd β-cycles are turned off if auxiliary (di)graph is too large.

Low Autocorrelated Binary Sequences

Problem: [used by Del Pia, Di Gregorio '21]

$$\min_{x \in \{-1,1\}^N} \sum_{i=1}^{N-R+1} \sum_{d=1}^{R-1} \left(\sum_{j=i}^{i+R-1-d} x_j x_{j+d} \right)^2 \quad \leftrightsquigarrow \quad \text{nonnegative degree-4 polynomial}$$

Parameters:

- $N \in \{10, 15, 20, \ldots, 50\}$ and $R \in \{\lceil \frac{k}{8} N \rceil : k = 1, 2, \ldots, 8\}$ ($9 \cdot 8 = 72$ instances)

	#solved	Closed gap mean	Running time geo.mean	Sepa time s.geo.mean	B&B nodes s.geo.mean
Old default (no k-flowers)	28	75 %	503.58 s	–	2026.39
1-flowers, no 2-flowers	30	90 %	410.96 s	2.15 s	640.28
New default (1- and 2-flowers)	29	89 %	419.40 s	3.15 s	651.42
Simple oβc's	26	84 %	[6 fails]	25.71 s	[6 fails]
Simple oβc's, delayed	27	88 %	550.20 s	116.67 s	269.37
Simple oβc's, delayed, only root	27	89 %	501.38 s	60.79 s	398.44
Rlxd oβc's	29	85 %	467.81 s	145.70 s	262.78
Rlxd oβc's, delayed	29	89 %	428.86 s	52.67 s	576.51
Rlxd oβc's, delayed, only root	30	89 %	417.87 s	14.61 s	619.18
Both oβc's, delayed	26	87 %	573.21 s	268.15 s	157.41

Unconstrained Random High-Degree

Problem: [used by Crama, Rodríguez-Heck '17]

N binaries, M monomials, degree $d \in \{m, m+1, m+2, \dots\}$ with probability $\approx 2^{m-d-1}$, coefs in $\mathcal{U}\{-10, 10\}$

Parameters:

- 5 instances per $N \in \{50, 100, 200\}$, $M \in \{5N, 10N, 20N\}$ and $m \in \{2, 3, 4\}$ $(5 \cdot 3 \cdot 3 \cdot 3 = 135$ instances)

	#solved	Closed gap mean	Running time geo.mean	Sepa time s.geo.mean	B&B nodes s.geo.mean
Old default (no k-flowers)	53	0.66 %	1041.98 s	–	4411.10
1-flowers, no 2-flowers	52	71 %	1011.22 s	0.59 s	2375.78
New default (1- and 2-flowers)	52	72 %	1027.10 s	1.76 s	1927.02
Simple oβc's	53	77 %	1206.02 s	369.70 s	145.70
Simple oβc's, delayed	54	78 %	1002.96 s	145.11 s	475.23
Simple oβc's, delayed, only root	52	78 %	982.39 s	96.42 s	435.67
Rlxd oβc's	50	70 %	1454.68 s	558.20 s	343.25
Rlxd oβc's, delayed	51	71 %	1220.32 s	190.24 s	1399.01
Rlxd oβc's, delayed, only root	51	72 %	1127.21 s	42.69 s	1629.70
Both oβc's, delayed	54	78 %	1067.71 s	240.03 s	401.20

Image Restoration

Problem: [used by Crama, Rodríguez-Heck '17]

Image restoration on w-by-h black/white image of which p pixels were perturbed; penalty per 2-by-2 patch

Parameters:

- 2 instances per $(w, h) \in \{(10,10), (10,15), (15,15), (15,20), (20,20), (20,25), (25,25),$ $p \in \{0, 5\,\%, 50\,\%\}$, three motifs (105 distinct instances)

	#solved	Closed gap mean	Running time geo.mean	Sepa time s.geo.mean	B&B nodes s.geo.mean
Old default (no k-flowers)	37	77 %	1596.01 s	–	1393.43
1-flowers, no 2-flowers	105	100 %	91.55 s	0.06 s	50.40
New default (1- and 2-flowers)	99	99 %	183.94 s	0.11 s	158.41
Simple oβc's	?	?	?	?	
Simple oβc's, delayed	101	99 %	74.58 s	7.56 s	5.96
Simple oβc's, delayed, only root	?	?	?	?	
Rlxd oβc's	78	92 %	328.84 s	133.20 s	150.36
Rlxd oβc's, delayed	97	99 %	220.01 s	19.30 s	148.93
Rlxd oβc's, delayed, only root	97	99 %	217.95 s	14.07 s	149.93
Both oβc's, delayed	?	?	?	?	

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Both oβc's, delayed	?	?	?

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Thank you! – Questions?