Extended Formulations for Radial Cones

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Joint work with

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Optimization & Augmentation

Combinatorial optimization problem:

- Ground set E (finite)
- Feasible solutions $\mathcal{F} \subseteq 2^E$
- Objective vector $c \in \mathbb{Q}^E$
- Goal: minimize cost $c(F) := \sum_{e \in F} c_e$ over all $F \in \mathcal{F}$.

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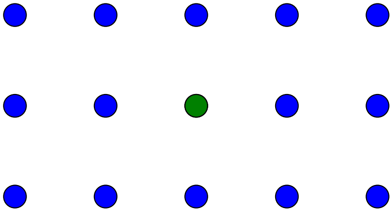
Theorem (Schulz, Weismantel & Ziegler, 1995; Grötschel & Lovász, 1995)

We can solve the augmentation problem (for arbitrary objective vectors) in polynomial time if and only if we can solve the optimization problem (for arbitrary objective vectors) in polynomial time.

Polyhedral Approach: Optimization

Polyhedral method:

- Identify $F \in \mathcal{F}$ with $\chi(F) \in \{0,1\}^E$ s.t. $\chi(F)_e = 1 \iff e \in F$.



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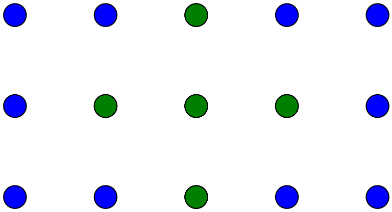
- Optimization problem is then to minimize $\langle c, x \rangle$ over $x \in X$.



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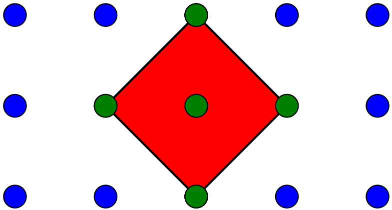

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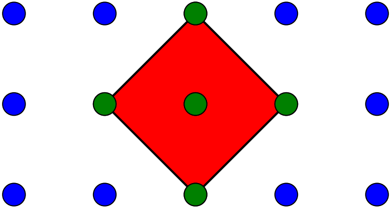
- Optimization problem is then to minimize $\langle c, x \rangle$ over $x \in \text{conv}(X)$.
- Find an outer description of $\text{conv}(X)$, i.e., $\text{conv}(X) = \left\{x \in \mathbb{R}^E : Ax \leq b\right\}$.
- Optimization problem is now an LP and we can use black-box solvers.



One drawback of the polyhedral method:

- Consider $X := \{x \in \{0,1\}^n : \sum_{i=1}^n \text{ even}\}$.
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Extended Formulations

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- Inequality description (Jeroslow, 1975) requires 2^{n-1} inequalities:

$$\sum_{i \in I}(1 - x_i) + \sum_{i \notin I} x_i \geq 1 \text{ for all } I \subseteq [n] \text{ with } |I| \text{ odd}$$

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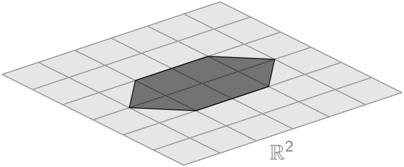
Potential cure: extended formulations

- $P = \text{conv}(X)$ has many facets, but maybe there exists an extension (Q, π) ($Q \subseteq \mathbb{R}^d$ polyhedron, $\pi : \mathbb{R}^d \to \mathbb{R}^n$ linear with $P = \pi(Q)$) with few facets?



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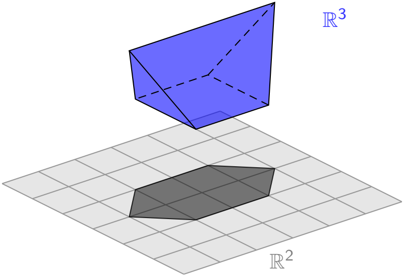

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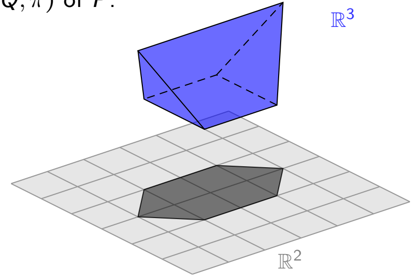

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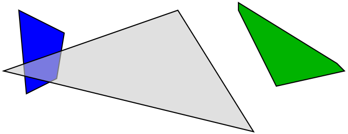
Let $P_1, \dots, P_k \subseteq \mathbb{R}^n$ be polytopes. Then $\text{xc}(\text{conv}(P_1 \cup \dots \cup P_k)) \leq \sum_{i=1}^k (\text{xc}(P_i) + 1)$.

Disjunctive programming:



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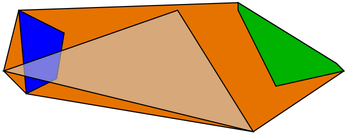

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- Applying the theorem: $\text{xc}(\text{conv}(X)) \leq \mathcal{O}(n^2)$

Limitations

Hard problems:

- Max-Cut problem: cut polytope for K_n (complete graph with n nodes) has extension complexity $2^{\Omega(n)}$ (Fiorini, Massar, Pokutta, Tiwary & de Wolf, 2012), best bound is 1.5^n (Kaibel & Weltge, 2013).

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- A perfect matching in a graph $G = (V, E)$ is a set $M \subseteq E$ with $|M \cap \delta(v)| = 1$.
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Theorem (Rothvoss, 2013)

For every even n, $\text{xc}(P_{pmatch}(n)) \geq 2^{\Omega(n)}$. Here, $P_{pmatch}(n)$ denotes the perfect matching polytope for K_n.

Polyhedral version of the augmentation problem:

- Consider a polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ and an objective vector $c \in \mathbb{R}^n$.
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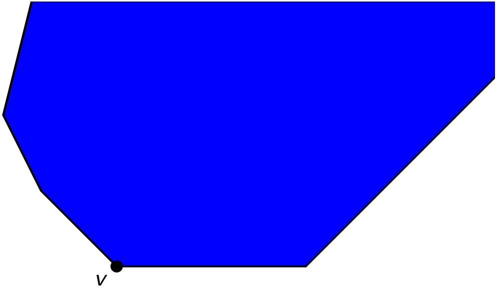
- The polyhedron for this task is the radial cone:

$$\begin{aligned} K_P(v) &:= \text{cone}(P - v) + v \\ &= \{x \in \mathbb{R}^n : A_{i,*}x \leq b_i \text{ for all } i \text{ with } A_{*,i}v = b_i\} \end{aligned}$$



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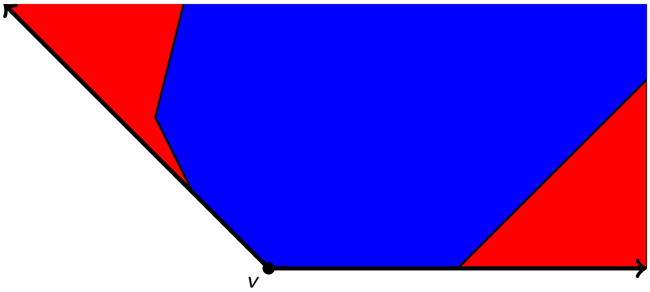

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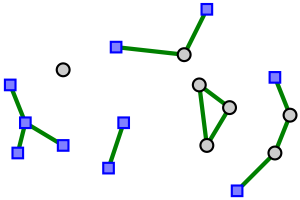
What remains?

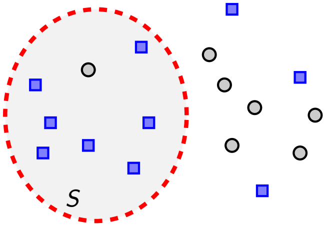
- Matching polytopes & friends (this talk).
- Stable-set polytopes of claw-free or perfect graphs.

T-Joins & T-Cuts

Definitions ($K_n = (V_n, E_n)$ complete graph on n nodes; $T \subseteq V$, $|T|$ even)**:**

- $J \subseteq E$ is a T-join if $|J \cap \delta(v)|$ is odd $\iff v \in T$
- $C = \delta(S) \subseteq E$ is a T-cut if $|S \cap T|$ is odd.





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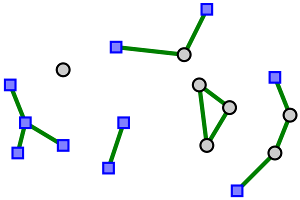

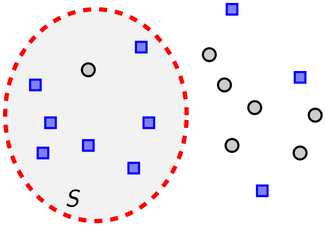

Facts:

- Both minimization problems can be solved in polynomial time for $c \geq \mathbb{O}$.
- Each T-join J intersects each T-cut C in at least one edge:

$$|J \cap C| = \langle \chi(J), \chi(C) \rangle \geq 1$$

T-Join- and T-Cut-Polyhedra

Polyhedra (Edmonds & Johnson, 1973)**:**

- T-join Polyhedron $P_{T\text{-join}}(n)^{\uparrow}$:

$$\begin{aligned} \langle \chi(C), x \rangle &\geq 1 \quad \text{for each } T\text{-cut } C \\ x_e &\geq 0 \quad \text{for each } e \in E \end{aligned}$$

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Relation to perfect matchings:

- A T-join $J \subseteq E$ is a perfect matching on nodes T if and only if $x = \chi(J)$ satisfies the valid inequalities $x_e \geq 0$ for all $e \in E \setminus E[T]$ and $\sum_{e \in \delta(v)} x_e \geq 1$ for all $v \in T$ with equality.

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Proposition (Walter & Weltge, 2018)

For every n and every set $T \subseteq V_n$, $\text{xc}(P_{T\text{-join}}(n)^{\uparrow}) \leq \mathcal{O}(n^2 \cdot 2^{|T|})$.

Blocking Polarity: Basics

Definitions:

- A polyhedron $P \subseteq \mathbb{R}^d_+$ is blocking if $x' \geq x$ implies $x' \in P$ for all $x \in P$.
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$$P = \{x \in \mathbb{R}^d_+ : \langle y^{(i)}, x \rangle \geq 1 \text{ for } i = 1, \dots, m\} \qquad (y^{(1)}, \dots, y^{(m)} \in \mathbb{R}^d_+)$$

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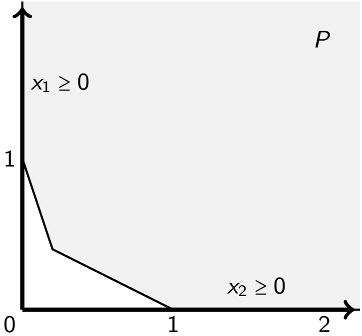

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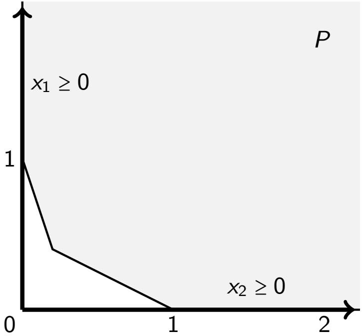

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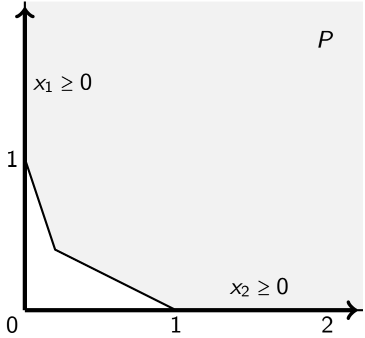

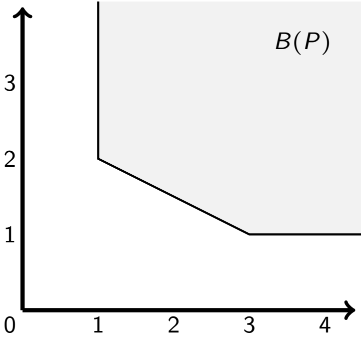

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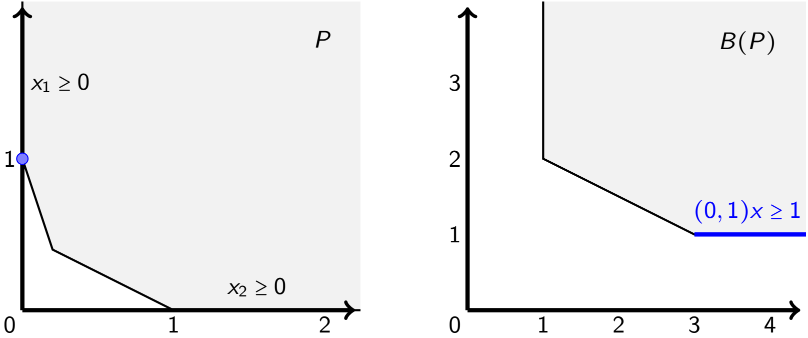

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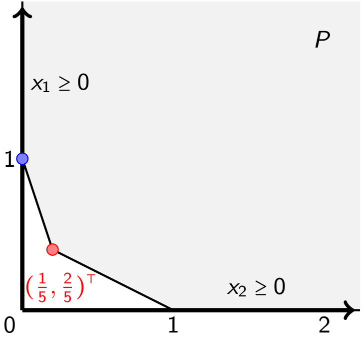

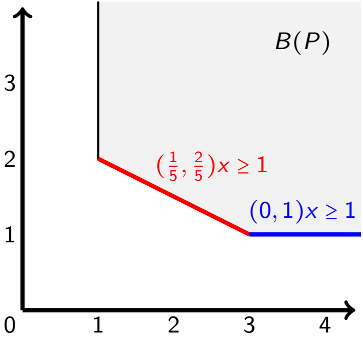

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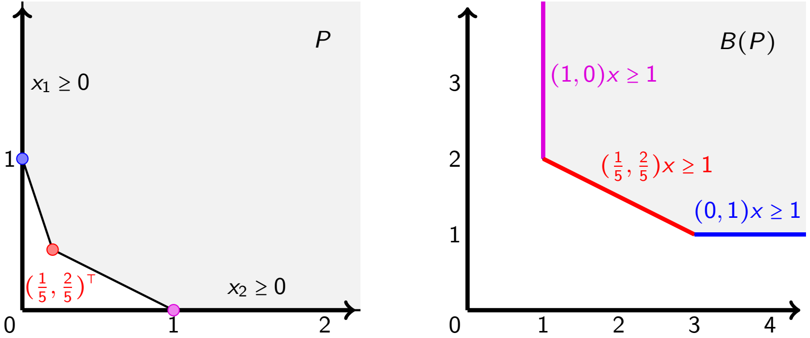

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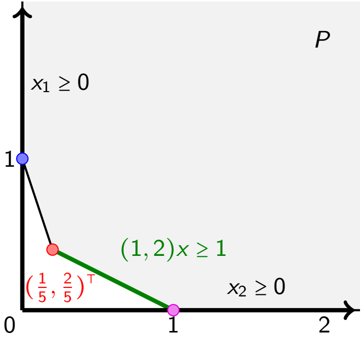

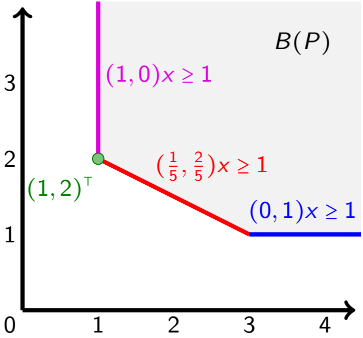

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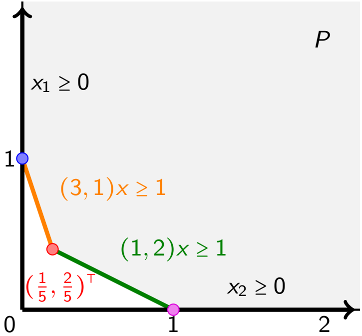

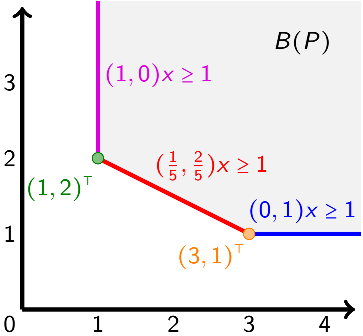

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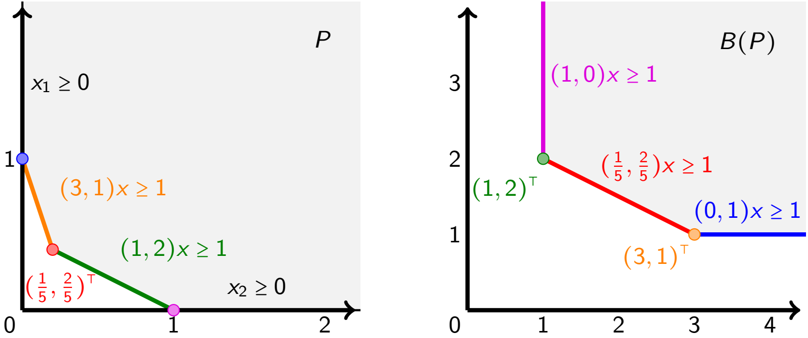

Blocking Polarity: Extensions

Proposition (Martin, 1991; Conforti, Kaibel, Walter & Weltge, 2015)

Given a non-empty polyhedron Q and $\gamma \in \mathbb{R}$, let

$$P := \{x : \langle y, x \rangle \geq \gamma \text{ for all } y \in Q\}.$$

Then $\operatorname{xc}(P) \leq \operatorname{xc}(Q) + 1$.

Proof idea:

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- $2^{\Omega(|T|)} \leq \text{xc}(P_{T\text{-cut}}(n)^{\uparrow}) \leq \mathcal{O}(n^2 \cdot 2^{|T|})$.

Blocking Polarity: Radial Cones

Polar object of radial cone:

- Any $v \in P$ defines a face $F_{B(P)}(v) := \{y \in B(P) : \langle v, y \rangle = 1\}$ of $B(P)$.

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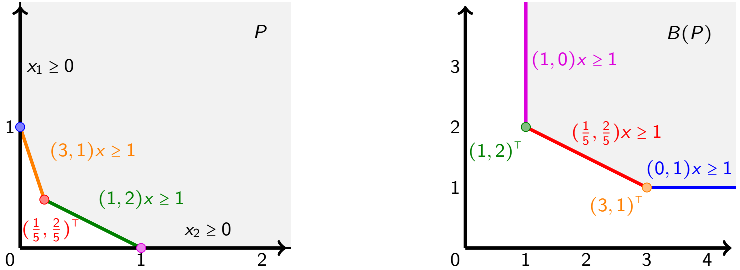
(i) $F_{B(P)}(v) = \left\{ y \in \mathbb{R}^d : \langle v, y \rangle = 1,\ \langle x, y \rangle \geq 1\ \forall x \in K_P(v) \right\}$.

(ii) $K_P(v) = \left\{ x \in \mathbb{R}^d : \langle y, x \rangle \geq 1\ \forall y \in F_{B(P)}(v) \right\}$.



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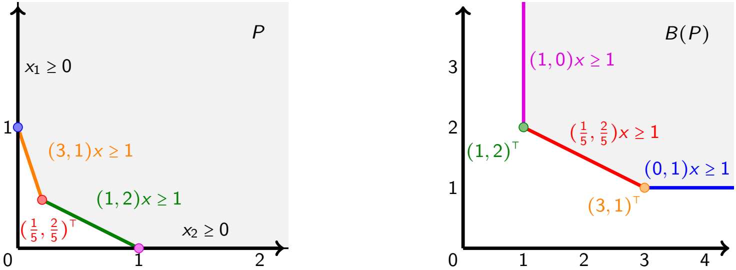

Consequence:

- $\mathrm{xc}(K_P(v))$ and $\mathrm{xc}(F_{B(P)}(v))$ differ by at most 1.
- To prove lower or upper bounds on $\mathrm{xc}(K_P(v))$, analyze $F_{B(P)}(v)$!

Radial Cones of T-Join Polyhedra

Theorem (Ventura & Eisenbrand, 2003)

For every set $T \subseteq V_n$ with $|T|$ even and every vertex v of $P_{T\text{-join}}(n)^{\uparrow}$, corresponding to a T-join $J \subseteq E_n$ in K_n, the extension complexity of the radial cone of $P_{T\text{-join}}(n)$ at v is most $\mathcal{O}(|J| \cdot n^2)$.

Their proof: ad-hoc construction using sets of flow variables.

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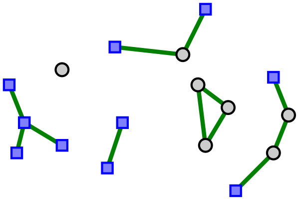

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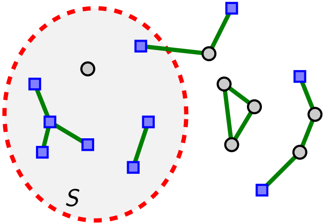

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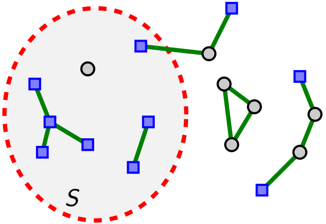

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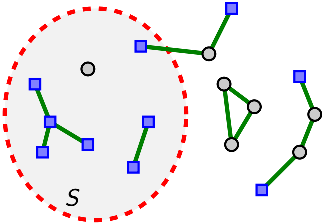

Theorem (Ventura & Eisenbrand, 2003)

For every set $T \subseteq V_n$ with $|T|$ even and every vertex v of $P_{T\text{-join}}(n)^{\uparrow}$, corresponding to a T-join $J \subseteq E_n$ in K_n, the extension complexity of the radial cone of $P_{T\text{-join}}(n)$ at v is most $\mathcal{O}(|J| \cdot n^2)$.

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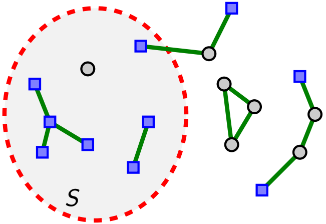

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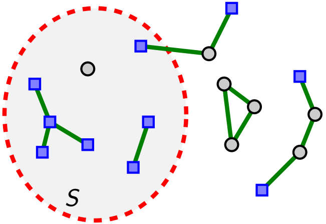

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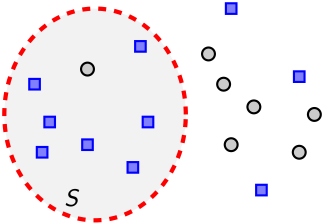

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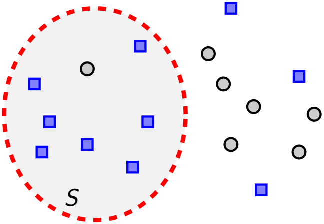

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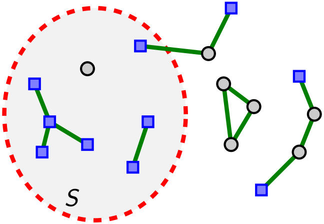

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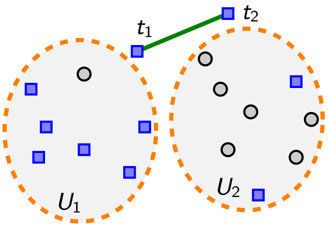

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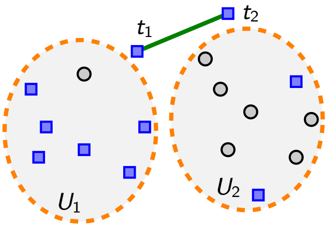

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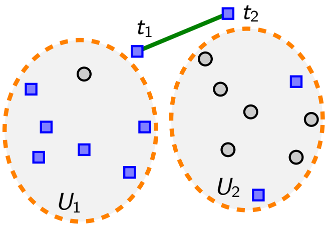

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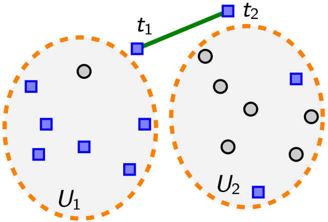

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Other candidates for investigation:

- Stable-set polytopes of claw-free graphs (current work with Gianpaolo Oriolo and Gautier Stauffer).
- Stable-set polytopes of perfect graphs (polyhedral description is known, but best (known) extended formulation has $\mathcal{O}(n^{\log n})$ facets).