UNIVERSITY OF TWENTE.

Common misbeliefs when working with total unimodularity

Matthias Walter

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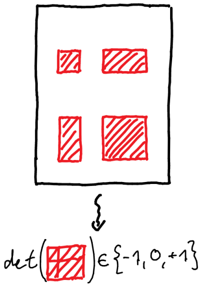
Aussois Combinatorial Optimization Workshop 2025

Definition – Total unimodularity

A matrix is **totally unimodular (TU)** if every square submatrix has determinant -1, 0 or $+1$.

Submatrices:



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Proposition – TU-preserving operations

Total unimodularity is maintained under these operations:

1. Transposition
2. Permutation of rows or columns
3. Scaling rows or columns by -1.
4. Taking submatrices
5. Appending copies of rows or columns.
6. Appending unit rows or columns
7. **Pivoting**

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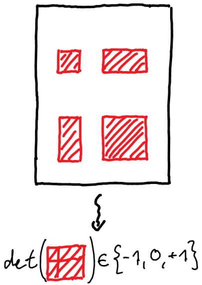

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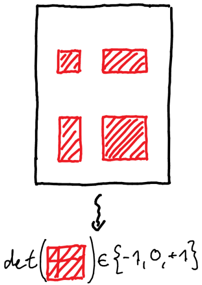

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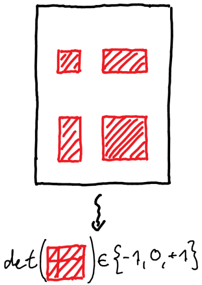

What again was pivoting?

- There was this thing called **pivot element** in Gaussian Elimination...
- So let's do elementary row operations: $\begin{bmatrix} -1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \xrightarrow{\text{add row 1 to row 2}} \begin{bmatrix} -1 & 0 \\ -1 & 1 \\ 1 & 1 \end{bmatrix}$

Pivots

Observation 1 – Pivoting $\neq$ elementary row operations

TU is not preserved by elementary row operations.

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Definition – Pivot **[§1.3 in Schrijver's '86 book]**

A **pivot** on the top-left nonzero α of a matrix $A = \begin{bmatrix} \alpha & b^\intercal \\ c & D \end{bmatrix}$ yields the matrix $A' = \begin{bmatrix} -\alpha^{-1} & \alpha^{-1} b^\intercal \\ \alpha^{-1} c & D - \alpha^{-1} c b^\intercal \end{bmatrix}$.

- **Question:** What is that formula about?

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$$\begin{bmatrix} \mathbb{I} & A \end{bmatrix} = \begin{bmatrix} 1 & \mathbb{O}^\intercal & \alpha & b^\intercal \\ \mathbb{O} & \mathbb{I} & c & D \end{bmatrix} \xrightarrow{\text{scale row 1}} \begin{bmatrix} \alpha^{-1} & \mathbb{O}^\intercal & 1 & \alpha^{-1} b^\intercal \\ \mathbb{O} & \mathbb{I} & c & D \end{bmatrix} \xrightarrow{\text{subtract row 1 from others}}$$

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Relevance for Polyhedral Combinatorics

Line of arguments – Integer rhs + TU matrix = integer polyhedron

Let A be **TU** and let b be an **integer** vector. Then $\{x \mid Ax \leq b\}$ is **integral**.

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A matrix A is **unimodular** if every square submatrix of order $rank(A)$ has determinant -1, 0 or $+1$.

Theorem – Integer rhs + unimodular matrix = integral polyhedra [Hoffman & Kruskal '56]

A matrix A is **unimodular** if and only if ...

- ... $\{x \mid Ax \leq b\}$ is **integral** for **every** integer right-hand side vector b.
- ... $\{x \mid Ax = b,\ x \geq \mathbb{O}\}$ is **integral** for **every** integer right-hand side vector b.

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Theorem – Integer rhs + TU matrix + integer bounds = integral polyhedra [Hoffman & Kruskal '56]

A matrix A is **TU** if and only if ...

- ... $\{x \mid Ax \leq b,\ \ell \leq x \leq u\}$ is **integral** for **every** integer right-hand side b and integer bounds ℓ, u.
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Regular matroids

Folklore – TU and regularity of matroids

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Let $v_1, v_2, \ldots, v_k \in \mathcal{F}^n$ for a field $\mathcal{F}$. The corresponding **linear matroid** $\mathcal{M}$ has ground set $\{v_1, v_2, \ldots, v_k\}$ and a subset is independent in $\mathcal{M}$ if it is linearly independent over $\mathcal{F}$.

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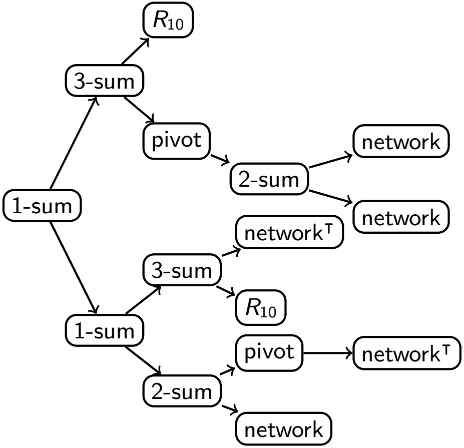
Proposition – TU and regularity of matroids

A matrix A is totally unimodular if and only the columns of $\begin{bmatrix} \mathbb{I} & A \end{bmatrix}$ define a matroid $\mathcal{M}$ that is a **linear matroid over every field**.

Seymour decomposition

Theorem – Seymour's decomposition theorem **[Seymour '80]**

A matrix is totally unimodular **if and only if** it arises from **network matrices** and copies of a particular 5-by-5-matrix R_{10} by TU-preserving operations and via ***k*-sums** for $k = 1, 2, 3$.



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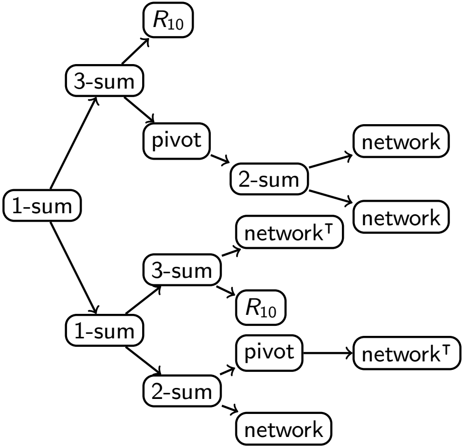
Exploited by several works:

- Seymour '80: *Decomposition of regular matroids.*
- Kashyap '08: *A decomposition theory for binary linear codes.*
- Dinitz & Kortsarz '12. *Matroid Secretary for Regular and Decomposable Matroids.*
- Artmann, Weismantel & Zenklusen '17: *A strongly polynomial algorithm for bimodular integer linear programming.*
- Aprile & Fiorini '21: *Regular Matroids Have Polynomial Extension Complexity.*
- Nägele, Santiago & Zenklusen '22: *Congruency-Constrained TU Problems Beyond the Bimodular Case.*
- Nägele, Nöbel, Santiago & Zenklusen 24'. *Advances on strictly* Δ*-modular IPs.*



Network matrices

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Theorem – TU of node-arc incidence matrices

The coefficient matrix of flow balance constraints (1) of any digraph $D = (V, A)$ is (totally) unimodular.

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Node-arc incidence matrices are a strict subset of network matrices!

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Definition – Network matrices **[Tutte '65]**

Let $D = (V, A)$ be a directed graph and let $T = (V, B)$ be a directed tree on V. The **network matrix** of (D, T) is the matrix $M \in \{-1, 0, +1\}^{B \times A}$ whose entries $M_{a,(v,w)}$ are

- $+1$ if the unique v-w-path in T passes through a forwardly,
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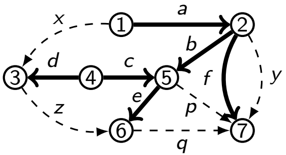
- and 0 otherwise.

Digraph with directed tree:



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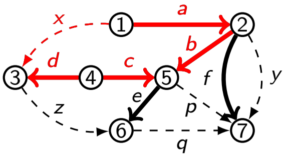

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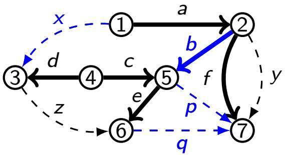

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Combining TU matrices: k-sums

1-sum

$$A \oplus_1 B := \begin{bmatrix} A & \mathbb{O} \\ \mathbb{O} & B \end{bmatrix}$$

2-sum

$$\begin{bmatrix} A & a \end{bmatrix} \oplus_2 \begin{bmatrix} b^\intercal \\ B \end{bmatrix} := \begin{bmatrix} A & ab^\intercal \\ \mathbb{O} & B \end{bmatrix}$$

3-sum

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Total unimodularity is preserved under 1-, 2- and 3-sum compositions.

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TU is **not** preserved under 3-sum **decomposition**.

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More flexible 3-sum definition:

$$\begin{bmatrix} A & a & a \\ c^\intercal & 0 & \varepsilon \end{bmatrix} \oplus_3 \begin{bmatrix} \varepsilon & 0 & b^\intercal \\ d & d & B \end{bmatrix} := \begin{bmatrix} A & ab^\intercal \\ dc^\intercal & B \end{bmatrix} \quad \text{for any } \varepsilon \in \{-1, +1\}$$

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Catch:

- If we choose the right ε-value then TU is also preserved when decomposing.
- By negating a, b and ε we obtain another 3-sum. There always exists one with $\varepsilon = 1$.
- Presented in Schrijver's '86 book, but only in the proof for decomposition algorithm.

Recognition of matrix class:

- Unimodular / totally unimodular matrices
- Network matrices
- Equimodular matrices
- Balanced matrices (polytime is work in progress)

Recognition of (binary / ternary) representation matrices of:

- Regular matroids
- Graphic matroids
- Series-parallel matroids

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Future plans:

- Perfect matrices
- Consecutive 1s
- Max-flow-min-cut matroids
- More certificates (computing forbidden submatrices / minors)

Usage

Example:

	x	y	z	p	q
a	1	0	0	0	0
b	1	0	0	-1	-1
c	-1	0	1	0	0
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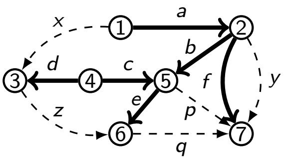
Graph:



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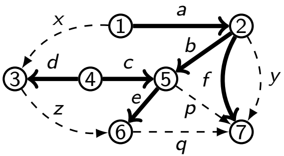


File input `demo.dense`:

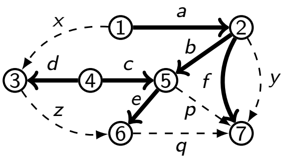
```
6 5

 1 0  0  0  0
 1 0  0 −1 −1
−1 0  1  0  0
 1 0 −1  0  0
 0 0  1  0 −1

 0 1  0  1  1
```

Usage

Example:

	x	y	z	p	q
a	1	0	0	0	0
b	1	0	0	-1	-1
c	-1	0	1	0	0
d	1	0	-1	0	0
e	0	0	1	0	-1
f	0	1	0	1	1

Graph:



File input `demo.dense`:

```
6 5

 1 0  0  0  0
 1 0  0 -1 -1
-1 0  1  0  0
 1 0 -1  0  0
 0 0  1  0 -1

 0 1  0  1  1
```

Invocation:

```
% cmr-network demo.dense -D demo.dot
Read 6x5 matrix with 13 nonzeros.
Matrix IS network.
Writing digraph to file <demo.dot>.

% dot -Tpdf demo.dot -o demo.pdf
```

Usage

Example:

	x	y	z	p	q
a	1	0	0	0	0
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d	1	0	-1	0	0
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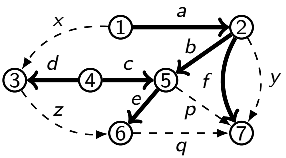
Graph:



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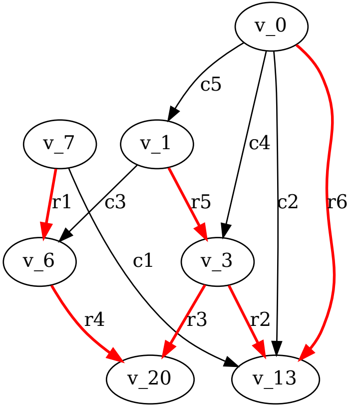

Usage

Example:

	x	y	z	p	q
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b	1	0	0	-1	-1
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d	1	0	-1	0	0
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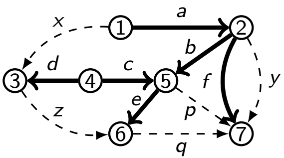

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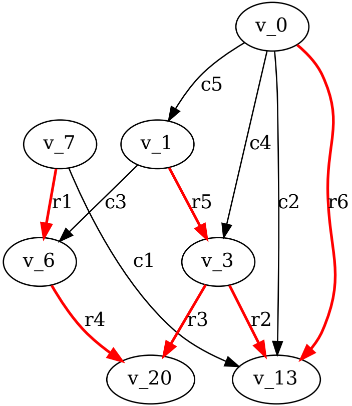

Coming soon: SageMath interface to play around with Seymour decompositions in **Python** [joint work with Luze Xu and Matthias Köppe (both UC Davis)]

New feature: Seymour decomposition

Requirement: Some algorithms rely on specific structure of decomposition!

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Theorem – Computed Seymour decomposition **[Walter '25]**

The recognition algorithm implemented in CMR returns (in polynomial time), for a given matrix M, either a submatrix N with $|\det(N)| = 2$ or a decomposition tree with the following properties:

- There are multiple valid R_{10} leaf nodes that arise by pivoting and scaling rows/columns by -1.
- No explicit scaling or transposition nodes are needed.
- Pivot nodes appear only as parents of 3-sum nodes.
- A 1-sum node can have more children and will be the root.
- Series-parallel reductions, a special kind of 2-sums, are treated directly and more efficiently.
- If a node's matrix is a (transposed) network matrix then the node is a (transposed) network node.
- Every leaf node has at least 3 rows and 3 columns.

New feature: Seymour decomposition

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- If a node's matrix is a (transposed) network matrix then the node is a (transposed) network node.
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Soon – Another 3-sum

Truemper defined an alternative 3-sum. There exists a decomposition tree with both 3-sum types but **no pivot** nodes.

Conclusions

Observation 1 – Pivoting $\neq$ elementary row operations

TU is not preserved by elementary row operations.

Observation 2 – Network matrices $\neq$ incidence matrices

Node-arc incidence matrices are a strict subset of network matrices!

Observation 3 – Be careful with 3-sums!

TU is **not** preserved under 3-sum **decomposition**, but there always exists a suitable one that preserves TU.

Observation 4 – TU only matters when you have additional variable bounds

Establish integrality of a polyhedron described by inequalities with integer right-hand side via **unimodularity**!
Use **total unimodularity** (only) if you have bounds!

Proposition – TU and regularity of matroids

A is TU if and only the columns of $\begin{bmatrix} \mathbb{I} & A \end{bmatrix}$ define a matroid that is a **linear matroid over every field**.

Thank you!