Integer Programming Methods
Homework set 3

Please conform to the following instructions:

1. Make the homework in groups of 2 or 3 persons.
2. Hand in you answers as pdf file.
   (a) Use the tile "`IntPM_Homework_Set_3_<groupname>.pdf`".
   (b) At the start of your file mention your names.
   (c) The report must be clearly written, concise and complete.
3. Hand in your report by e-mail to m.walter@utwente.nl.

Exercise 1

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be *laminar* families of subsets of some finite ground set E, i.e., for $i = 1, 2$, $\mathcal{L}_i \subseteq 2^E$ and $A, B \in \mathcal{L}_i$ implies that $A \cap B = \varnothing$ or $A \subseteq B$ or $B \subseteq A$. Prove that the incidence matrix $M = \begin{pmatrix} M^1 \\ M^2 \end{pmatrix}$ is totally unimodular, where M^i is the (binary) incidence matrix of $\mathcal{L}_i$ with one row for each $A \in \mathcal{L}_i$, one column for each $e \in E$ and with entry $M^i_{A,e} = 1$ if and only if $e \in A$.

Hint: It may be helpful to represent a laminar family $\mathcal{L}$ by a tree whose nodes are the sets $A \in \mathcal{L}$ and where an edge from A to B indicates that B is a maximal subset of A. The parity of the depth of each set may help you in the row version of the Ghouila-Houri criterion (Theorem 4.6)

Exercise 2

Let $D = (V, A)$ be a digraph. For every $a \in A$, let $\ell_a, u_a \in \mathbb{R}_{\geq 0}$ be given such that $\ell_a \leq u_a$. Show that the set of circulations $\{x \in \mathbb{R}^A : A_D x = 0,\ \ell \leq x \leq u\}$ (with A_D being the node-arc incidence matrix of D) is nonempty if and only if

$$\sum_{a \in \delta^-(X)} \ell_a \leq \sum_{a \in \delta^+(X)} u_a \text{ for all } X \subseteq V \tag{1}$$

holds, for instance via the following strategy:

1. Characterize nonemptyness via Farkas' Lemma (Theorem 3.4).
2. Augment the resulting system by (binary) box constraints on the variables to make it bounded.

3. Exploit total unimodularity of A_D to obtain a binary "Farkas certificate" and extract X.

Exercise 3

Let $G = (V, E)$ be a connected graph. The *cut polyhedron* of G is the convex hull of incidence vectors of all nontrivial cuts plus the first orthant, i.e., $P_{\text{mincut}}(G) := \text{conv}\{\chi(\delta(S)) \in \{0,1\}^E : \varnothing \neq S \subsetneq V\} + \mathbb{R}^E_+$.

Prove that $P_{\text{mincut}}(G)$ has an extended formulation of size $\mathcal{O}(|V| \cdot |E|)$. Proceed as follows.

(a) Consider, for distinct $s, t \in V$, the bidirected graph of G with unit capacities.

(b) Model the problem of finding an s-t-flow with flow value 1 as an LP.

(c) Dualize this LP, fix dual variables for s and t to 0 and 1 to derive an extended formulation for all arc sets that contain an s-t-cut. (Note that this is effectively the proof of Lemma 4.14 in the book.)

(d) Apply Theorem 4.39 by taking the union of projections of such polyhedra for fixed s and varying t.

Do not worry too much about the sum with $\mathbb{R}^E_+$. It will naturally appear in the dual of the flow LP.