

Integer Programming Methods

Homework set 4

Please conform to the following instructions:

1. Make the homework in groups of 2 or 3 persons.
 2. Hand in you answers as pdf file.
 - (a) Use the tile “IntPM.Homework_Set_4-<groupname>.pdf”.
 - (b) At the start of your file mention your names.
 - (c) The report must be clearly written, concise and complete.
 3. Hand in your report by e-mail to m.walter@utwente.nl.
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Exercise 1

Consider the Steiner tree problem for a graph $G = (V, E)$ with terminals $T \subseteq V$ and edge costs $c \in \mathbb{R}_{\geq 0}^E$. Let $D = (V, A)$ be the bidirected graph of G and let $r \in T$ be any root terminal. In the lecture we discussed the *undirected cut formulation*

$$\min \sum_{e \in E} c_e x_e \tag{1a}$$

$$\text{s.t.} \quad \sum_{e \in \delta(S)} x_e \geq 1 \quad \forall S \subseteq V : r \in S, T \setminus S \neq \emptyset \tag{1b}$$

$$x \in \mathbb{Z}_{\geq 0}^E \tag{1c}$$

in which variables x_e indicate whether edge $e \in E$ is bought ($x_e = 1$) or not ($x_e = 0$). Now consider the following *directed cut formulation* which exploits the following fact: every Steiner tree F can be “rooted away from r ”. This means, instead of the edge $\{u, v\} \in F$ we consider either the arc (u, v) or the arc (v, u) , such that the unique r - v -path visits node u . Such a subgraph of D is called a **Steiner arborescence**.

$$\min \sum_{e \in E} c_e x_e \tag{2a}$$

$$\text{s.t.} \quad \sum_{a \in \delta^{\text{out}}(S)} y_a \geq 1 \quad \forall S \subseteq V : r \in S, T \setminus S \neq \emptyset \tag{2b}$$

$$y_{(u,v)} + y_{(v,u)} \leq x_{\{u,v\}} \quad \forall \{u, v\} \in E \tag{2c}$$

$$y \in \mathbb{R}_{\geq 0}^A \tag{2d}$$

$$x \in \mathbb{Z}_{\geq 0}^E \tag{2e}$$

$$\tag{2f}$$

The goal of this exercise is to show that the directed formulation is strictly stronger than the undirected formulation (considering the projection on the x -variables).

1. Show that formulation (2) is always stronger than formulation (1).
2. Show that formulation (1) is *not* stronger than formulation (2). To this end, consider a triangle graph with all nodes being terminals and unit costs $c = (1, 1, 1)^\top$ and show that $x = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})^\top$ is feasible for the LP relaxation of (1) but not for the LP relaxation of (2).

Exercise 2

Construct a polyhedron P for which split cuts are stronger than Chvátal-Gomory cuts. To prove this, construct also an inequality $a^\top x \leq \beta$ that is a split cut but not implied by Chvátal-Gomory cuts.

Exercise 3

By Theorem 4.25 we know a perfect formulation for the minimum cost spanning tree problem on a graph $G' = (V', E')$, namely the spanning tree polytope

$$P_{\text{sptree}}(G') := \text{conv}\{x^T \in \{0, 1\}^{E'} : x^T \text{ incidence vector a spanning tree } T \text{ of } G'\} \quad (3)$$

is equal to

$$\{x \in \mathbb{R}_{\geq 0}^E : \sum_{e \in E'} x_e = |V'| - 1, \sum_{e \in E'[S]} x_e \leq |S| - 1 \forall \emptyset \neq S \subsetneq V'\} \quad (4)$$

holds. We consider the *Subtour formulation* for the Traveling Salesperson Problem for a graph $G = (V, E)$.

$$\min \sum_{e \in E} c_e x_e \quad (5a)$$

$$\text{s.t.} \quad \sum_{e \in \delta(v)} x_e = 2 \quad \forall v \in V \quad (5b)$$

$$\sum_{e \in E[S]} x_e \leq |S| - 1 \quad \forall S \subset V, 2 \leq |S| \leq |V| - 2 \quad (5c)$$

$$x_e \geq 0 \quad \forall e \in E \quad (5d)$$

Let $r \in V$ be an arbitrary node and let $G' = (V', E')$ with $V' := V \setminus \{r\}$ and $E' := E \setminus \delta(r)$ be the subgraph of G obtained by removing r . Your goal is to prove that (5c) can be replaced by requiring that the restriction of x on the edge set E' is in the spanning-tree polytope of G' , i.e., that the subtour relaxation of G is equal to the set of $x \in \mathbb{R}^E$ satisfying

$$\begin{aligned} \text{proj}_{E'}(x) &\in P_{\text{sptree}}(G') \\ \sum_{e \in \delta(v)} x_e &= 2 \quad \forall v \in V \\ x_e &\geq 0 \quad \forall e \in E, \end{aligned}$$

Remark: This implies that from any extended formulation of $P_{\text{sptree}}(G')$ of size k we can derive an extended formulation for the subtour relaxation of size $\mathcal{O}(k)$. In particular, there are such formulations with $k = \mathcal{O}(|V|)$ for planar graphs.