Integer Programming Methods
Homework set 4

Please conform to the following instructions:

1. Make the homework in groups of 2 or 3 persons.
2. Hand in you answers as pdf file.
   (a) Use the tile "`IntPM_Homework_Set_4_<groupname>.pdf`".
   (b) At the start of your file mention your names.
   (c) The report must be clearly written, concise and complete.
3. Hand in your report by e-mail to m.walter@utwente.nl.

Exercise 1

Consider the Steiner tree problem for a graph $G = (V, E)$ with terminals $T \subseteq V$ and edge costs $c \in \mathbb{R}^E_{\geq 0}$. Let $D = (V, A)$ be the bidirected graph of G and let $r \in T$ be any root terminal. In the lecture we discussed the *undirected cut formulation*

$$\min \sum_{e \in E} c_e x_e \tag{1a}$$

$$\text{s.t.} \sum_{e \in \delta(S)} x_e \geq 1 \qquad \forall S \subseteq V : r \in S,\ T \setminus S \neq \varnothing \tag{1b}$$

$$x \in \mathbb{Z}^E_{\geq 0} \tag{1c}$$

in which variables x_e indicate whether edge $e \in E$ is bought ($x_e = 1$) or not ($x_e = 0$). Now consider the following *directed cut formulation* which exploits the following fact: every Steiner tree F can be "rooted away from r". This means, instead of the edge $\{u, v\} \in F$ we consider either the arc (u, v) or the arc (v, u), such that the unique r-v-path visits node u. Such a subgraph of D is called a **Steiner arborescence**.

$$\min \sum_{e \in E} c_e x_e \tag{2a}$$

$$\text{s.t.} \sum_{a \in \delta^{\text{out}}(S)} y_a \geq 1 \qquad \forall S \subseteq V : r \in S,\ T \setminus S \neq \varnothing \tag{2b}$$

$$y_{(u,v)} + y_{(v,u)} \leq x_{\{u,v\}} \qquad \forall \{u, v\} \in E \tag{2c}$$

$$y \in \mathbb{R}^A_{\geq 0} \tag{2d}$$

$$x \in \mathbb{Z}^E_{\geq 0} \tag{2e}$$

$$\tag{2f}$$

The goal of this exercise is to show that the directed formulation is strictly stronger than the undirected formulation (considering the projection on the x-variables).

1. Show that formulation (2) is always stronger than formulation (1).
2. Show that formulation (1) is *not* stronger than formulation (2). To this end, consider a triangle graph with all nodes being terminals and unit costs $c = (1,1,1)^\intercal$ and show that $x = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})^\intercal$ is feasible for the LP relaxation of (1) but not for the LP relaxation of (2).

Exercise 2

Construct a polyhedron P for which split cuts are stronger than Chvátal-Gomory cuts. To prove this, construct also an inequality $a^\intercal x \leq \beta$ that is a split cut but not implied by Chvátal-Gomory cuts.

Exercise 3

By Theorem 4.25 we know a perfect formulation for the minimum cost spanning tree problem on a graph $G' = (V', E')$, namely the the spanning tree polytope

$$P_{\text{sptree}}(G') := \text{conv}\{x^T \in \{0,1\}^{E'} : x^T \text{ incidence vector a spanning tree } T \text{ of } G'\} \tag{3}$$

is equal to

$$\{x \in \mathbb{R}^E_{\geq 0} : \sum_{e \in E'} x_e = |V'| - 1, \quad \sum_{e \in E'[S]} x_e \leq |S| - 1 \; \forall \varnothing \neq S \subsetneq V'\} \tag{4}$$

holds. We consider the *Subtour formulation* for the Traveling Salesperson Problem for a graph $G = (V, E)$.

$$\begin{aligned}
\min \quad & \sum_{e \in E} c_e x_e && && \text{(5a)}\\
\text{s.t.} \quad & \sum_{e \in \delta(v)} x_e = 2 && \forall v \in V && \text{(5b)}\\
& \sum_{e \in E[S]} x_e \leq |S| - 1 && \forall S \subset V,\ 2 \leq |S| \leq |V| - 2 && \text{(5c)}\\
& x_e \geq 0 && \forall e \in E && \text{(5d)}
\end{aligned}$$

Let $r \in V$ be an arbitrary node and let $G' = (V', E')$ with $V' := V \setminus \{r\}$ and $E' := E \setminus \delta(r)$ be the subgraph of G obtained by removing r. Your goal is to prove that (5c) can be replaced by requiring that the restriction of x on the edge set E' is in the spanning-tree polytope of G', i.e., that the subtour relaxation of G is equal to the set of $x \in \mathbb{R}^E$ satisfying

$$\begin{aligned}
& \text{proj}_{E'}(x) \in P_{\text{sptree}}(G') \\
& \sum_{e \in \delta(v)} x_e = 2 && \forall v \in V \\
& x_e \geq 0 && \forall e \in E,
\end{aligned}$$

Remark: This implies that from any extended formulation of $P_{\text{sptree}}(G')$ of size k we can derive an extended formulation for the subtour relaxation of size $\mathcal{O}(k)$. In particular, there are such formulations with $k = \mathcal{O}(|V|)$ for planar graphs.