Integer Programming Methods
Week 2

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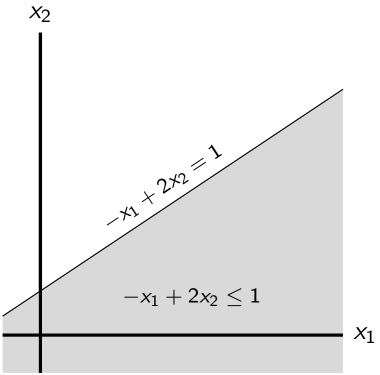
Polyhedra

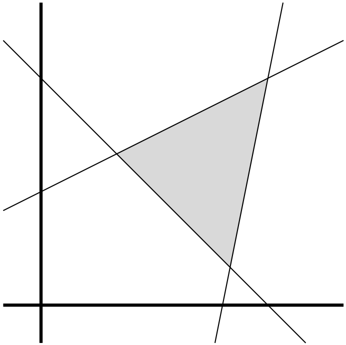
Halfspace



Definition

For a vector $\mathbf{a} \in \mathbb{R}^n$ and scalar $b \in \mathbb{R}$ the set $\{\mathbf{x} \in \mathbb{R}^n | \mathbf{a}^T \mathbf{x} \leq b\}$ is called a **halfspace**.

Polyhedron



Definition

The intersection of a finite number of halfspaces is a **polyhedron**.

Polyhedron

- Consider m halfspaces in $\mathbb{R}^n$, such that for $1 \leq i \leq m$ the halfspace H_i is defined as

$$H_i = \{\mathbf{x} \in \mathbb{R}^n | \mathbf{a}_i^T \mathbf{x} \leq b_i\}$$

- The polyhedron $P = \bigcap_{i=1}^m H_i$ can be represented as

$$\begin{aligned} P &= \bigcap_{i=1}^{m} \{\mathbf{x} \in \mathbb{R}^n | \mathbf{a}_i^T \mathbf{x} \leq b_i\} \\ &= \{\mathbf{x} \in \mathbb{R}^n | \mathbf{a}_1^T \mathbf{x} \leq b_1, \ldots, \mathbf{a}_m^T \mathbf{x} \leq b_m\} \end{aligned}$$

- Using the matrix $A = \begin{bmatrix} \mathbf{a}_1^T \\ \vdots \\ \mathbf{a}_m^T \end{bmatrix}$ and vector $\mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$ we can express P as

$$P = \{\mathbf{x} \in \mathbb{R}^n | A\mathbf{x} \leq \mathbf{b}\}$$

Polyhedron

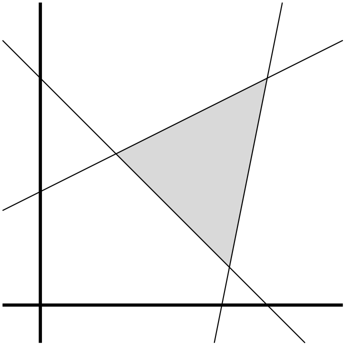
Below is an equivalent definition of a polyhedron:

Definition

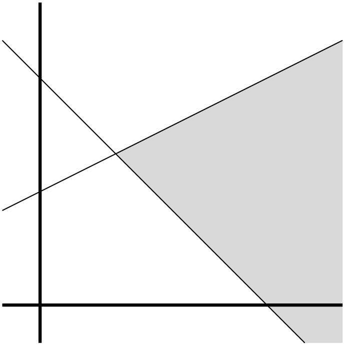
A set $P \subseteq \mathbb{R}^n$ is called a **polyhedron** if there exists some $m \times n$ matrix A and $\mathbf{b} \in \mathbb{R}^m$ such that

$$P = \{\mathbf{x} \in \mathbb{R}^n | A\mathbf{x} \leq \mathbf{b}\}$$

Boundedness



Bounded polyhedron (polytope)



Unbounded polyhedron

Convexity

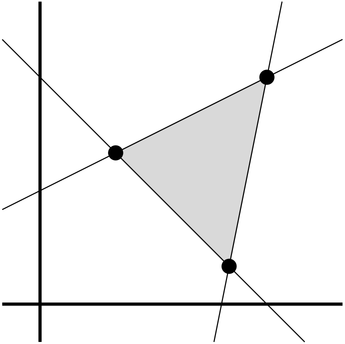
Theorem

Every polyhedron is a convex set.

Remember:
Recall that this means that for a polyhedron P, $\mathbf{x}_1 \in P$, $\mathbf{x}_2 \in P$ and $\lambda \in [0, 1]$ it follows for $\mathbf{x} = \lambda \mathbf{x}_1 + (1 - \lambda)\mathbf{x}_2$ that $\mathbf{x} \in P$.

Extreme points

Extreme points



Definition

Let P be a polyhedron. The point $\mathbf{x}$ is called an **extreme point** (or **vertex**) of P if $\mathbf{x} \in P$ and there are no two distinct points $\mathbf{x}_1, \mathbf{x}_2 \in P$ such that

$$\mathbf{x} = \lambda \mathbf{x}_1 + (1 - \lambda)\mathbf{x}_2 \text{ for some } \lambda \in (0, 1)$$

Properties of extreme points

Theorem

For $A \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$ consider the polyhedron $P = \{\mathbf{x} \in \mathbb{R}^n | A\mathbf{x} \leq \mathbf{b}\}$. The point $\mathbf{x} \in P$ is an extreme point if and only if there is equality in at least n linearly independent rows of the inequalities $A\mathbf{x} \leq \mathbf{b}$, (i.e., the corresponding rows of A are linearly independent).

Observations:

- If $m < n$ there cannot be any extreme points.
- If $P = \{\mathbf{x} \in \mathbb{R}^n | A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$, the nonnegativity constraints should be seen as '*rows of the inequality*', so $m + n$ inequalities in total in this case.
- If $A \in \mathbb{R}^{m \times n}$, $P = \{\mathbf{x} \in \mathbb{R}^n | A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ and $m < n$, every extreme point $\mathbf{x}$ of P has at least $n - m$ zero elements.

Properties of extreme points

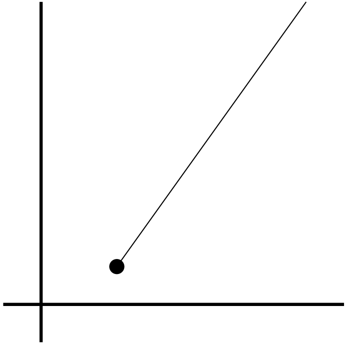
Corollary

For any polyhedron P the set of extreme points is **finite**.

Note:
In theory we could write down all extreme points of a polyhedron. However, there are in the worst case $\binom{m}{n}$ different extreme points.

Extreme directions

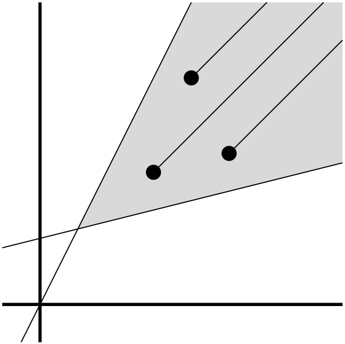
Ray



Definition

Consider the **point** $\mathbf{x}_0 \in \mathbb{R}^n$ and **direction** $\mathbf{d} \in \mathbb{R}^n$. The set $\{\mathbf{x} \in \mathbb{R}^n | \mathbf{x} = \mathbf{x}_0 + \mu\mathbf{d}, \mu \geq 0\}$ is called a **ray** (or **half-line**).

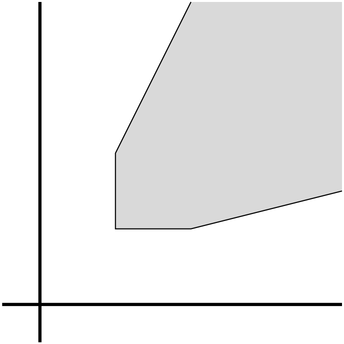
Direction of a polyhedron



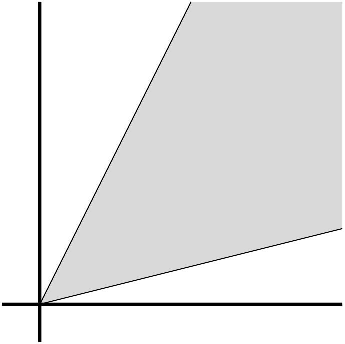
Definition

The vector $\mathbf{d} \in \mathbb{R}^n$ is called a **direction** of the polyhedron P (or convex set in general) if $\mathbf{d} \neq \mathbf{0}$ and for every point $\mathbf{x}_0 \in P$ it holds that the corresponding ray is included in the polyhedron, that is, $\{\mathbf{x} \in \mathbb{R}^n | \mathbf{x} = \mathbf{x}_0 + \mu\mathbf{d}, \mu \geq 0\} \subseteq P$.

Recession cone



Polyhedron



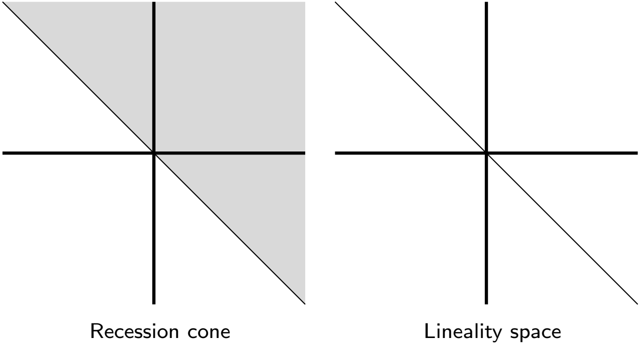
Recession cone

Definition

The set of all directions of a polyhedron P is called the **recession cone** of P, denoted by $\text{rec}(P)$. That is,

$$\text{rec}(P) = \{\mathbf{d} \in \mathbb{R}^n : \mathbf{x}_0 + \mu\mathbf{d} \in P, \forall \mathbf{x}_0 \in P, \mu \geq 0\}.$$

Lineality space



Definition

The **lineality space** of P, denoted by $\text{lin}(P)$ is,

$$\text{lin}(P) = \{\mathbf{d} \in \mathbb{R}^n : \mathbf{x}_0 + \mu \mathbf{d} \in P, \forall \mathbf{x}_0 \in P, \mu \in \mathbb{R}\}.$$

Properties

Theorem

For $A \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$ consider the polyhedron $P = \{\mathbf{x} \in \mathbb{R}^n | A\mathbf{x} \leq \mathbf{b}\}$. It holds that

$$rec(P) = \{\mathbf{d} \in \mathbb{R}^n : A\mathbf{d} \leq \mathbf{0}\}$$

and

$$lin(P) = \{\mathbf{d} \in \mathbb{R}^n : A\mathbf{d} = \mathbf{0}\}.$$

Direction of a polyhedron

Proof.

For a direction $\mathbf{d}$ of P, it holds for all $\mathbf{x}_0 \in P$ and $\mu \geq 0$ that $A(\mathbf{x}_0 + \mu\mathbf{d}) \leq \mathbf{b}$. Therefore, we derive that $A\mathbf{x}_0 + A\mu\mathbf{d} \leq \mathbf{b}$, or equivalently $\mu(A\mathbf{d}) \leq \mathbf{b} - A\mathbf{x}_0$. Since this must hold for all $\mu \geq 0$, it must be true that $A\mathbf{d} \leq \mathbf{0}$. Hence, $\text{rec}(P) \subseteq \{\mathbf{d} \in \mathbb{R}^n : A\mathbf{d} \leq \mathbf{0}\}$.

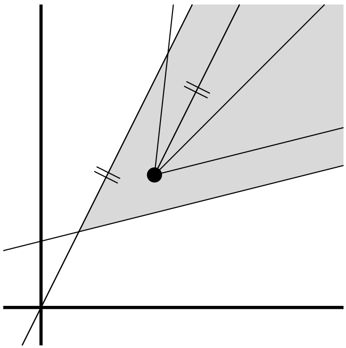
For the reverse, observe that if $A\mathbf{d} \leq \mathbf{0}$, then for all $\mathbf{x}_0 \in P$ and $\mu \geq 0$, it holds that $A(x_0 + \mu\mathbf{d}) \leq \mathbf{b}$. We conclude $\{\mathbf{d} \in \mathbb{R}^n : A\mathbf{d} \leq 0\} \subseteq \text{rec}(P)$, and it follows that $\text{rec}(P) = \{\mathbf{d} \in \mathbb{R}^n : A\mathbf{d} \leq \mathbf{0}\}$.

Observe that $\text{lin}(P) = \text{rec}(P) \bigcap -\text{rec}(P)$. Hence $\text{lin}(P) = \{\mathbf{d} \in \mathbb{R}^n : A\mathbf{d} = \mathbf{0}\}$. □

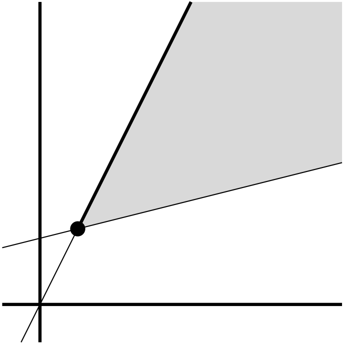
Extreme directions of a polyhedron

Definition

A direction $\mathbf{d}$ of P is called an **extreme direction** if there are no two directions $\mathbf{d}_1$ and $\mathbf{d}_2$ of P (such that $\mu_1\mathbf{d}_1 \neq \mathbf{d}$ and $\mu_2\mathbf{d}_2 \neq \mathbf{d}$ for all $\mu_1, \mu_2 \geq 0$) for which it holds that there exist $\lambda_1, \lambda_2 > 0$ so that $\mathbf{d} = \lambda_1\mathbf{d}_1 + \lambda_2\mathbf{d}_2$.



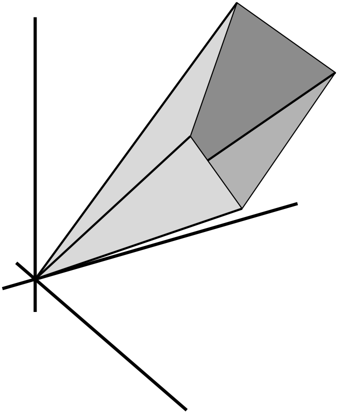
Extreme ray



The ray $\{\mathbf{x} \in \mathbb{R}^n | \mathbf{x} = \mathbf{x}_0 + \mu \mathbf{d}, \mu \geq 0\}$, where $\mathbf{x}_0 \in P$ is an extreme point of P and $\mathbf{d}$ is a "*corresponding*" extreme direction of P, is usually called an **extreme ray**.

Properties of extreme directions

- If $\mathbf{d}$ is a(n extreme) direction of P then so is $\lambda\mathbf{d}$ for $\lambda > 0$.
- A cross section of rec(P) '*represents*' all directions.



Properties of extreme directions

Theorem

Let us say that a direction $\mathbf{d}$ *is normalized if* $\mathbf{e}^T\mathbf{d} = 1$. *A normalized direction* $\mathbf{d}^*$ *of* $P = \{\mathbf{x} \in \mathbb{R}^n | A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ *is an extreme direction if and only if* $\mathbf{d}^*$ *is an extreme point of the polyhedron*

$$P' = \{\mathbf{d} \in \mathbb{R}^n | A\mathbf{d} \leq \mathbf{0}, \mathbf{e}^T\mathbf{d} = 1, \mathbf{d} \geq \mathbf{0}\}$$

Properties of extreme directions

Proof.

First prove that if $\mathbf{d}^*$ is a normalized extreme direction of P then it is an extreme point of P':
(Proof by contradiction)

- Suppose $\mathbf{d}^*$ is an extreme direction of P but not an extreme point of P'.
- By a previous theorem it follows that $A\mathbf{d}^* \leq \mathbf{0}$ and $\mathbf{d}^* \geq \mathbf{0}$, moreover we assume $\mathbf{e}^T\mathbf{d}^* = 1$, hence $\mathbf{d}^* \in P'$.
- Because we assume $\mathbf{d}^*$ is not an extreme point of P', there exist points $\mathbf{d}_1, \mathbf{d}_2$ in P' and $\lambda \in (0, 1)$ such that

$$\mathbf{d}^* = \lambda\mathbf{d}_1 + (1-\lambda)\mathbf{d}_2$$

...

Properties of extreme directions

Proof continued.

- Consider the point $\mathbf{x}_0 \in P$ and scalar $\mu \geq 0$. We derive

$$\begin{aligned} A\left(\mathbf{x}_0 + \mu \mathbf{d}_1\right) &= A\mathbf{x}_0 + \mu A\mathbf{d}_1 \\ &\leq \mathbf{b} + \mu \mathbf{0} \\ &= \mathbf{b} \end{aligned}$$

- Also $\mathbf{x}_0 + \mu \mathbf{d}_1 \geq \mathbf{0}$.
- As this holds for all $\mu \geq 0$, by definition of a direction $\mathbf{d}_1$ must be a direction of P.
- Similarly $\mathbf{d}_2$ is a direction of P.
- Because $\mathbf{d}_1$ and $\mathbf{d}_2$ are directions of P and $\mathbf{d}^* = \lambda \mathbf{d}_1 + (1-\lambda)\mathbf{d}_2$, it follows that $\mathbf{d}^*$ cannot be an extreme direction of P which contradicts our assumption.

...

Properties of extreme directions

Proof continued.

Finally prove that if $\mathbf{d}^*$ is an extreme point of P' then it is a normalized extreme direction of P:
(Proof by contradiction)

- Suppose $\mathbf{d}^*$ is an extreme point of P' but not an extreme direction of P.
- Since $A\mathbf{d}^* \leq \mathbf{0}$ and $\mathbf{d} \geq \mathbf{0}$, it follows for all $\mathbf{x} \in P$ and $\mu \geq 0$ that $A(\mathbf{x} + \mu\mathbf{d}^*) \leq \mathbf{b}$ and $\mathbf{x} + \mu\mathbf{d}^* \geq \mathbf{0}$, hence $\mathbf{d}^*$ is a direction of P.
- Because we assume $\mathbf{d}^*$ is not an extreme direction of P, there exist directions $\mathbf{d}_1, \mathbf{d}_2$ of P (such that $\mu_1\mathbf{d}_1 \neq \mathbf{d}$ and $\mu_2\mathbf{d}_2 \neq \mathbf{d}$ for all $\mu_1, \mu_2 \geq 0$) and $\lambda_1, \lambda_2 > 0$ such that

$$\mathbf{d}^* = \lambda_1\mathbf{d}_1 + \lambda_2\mathbf{d}_2$$

- Without loss of generality we may assume $\mathbf{e}^T\mathbf{d}_1 = \mathbf{e}^T\mathbf{d}_2 = 1$.

...

Properties of extreme directions

Proof continued.

- Observe that

$$\begin{aligned} 1 &= \mathbf{e}^T\mathbf{d}^* \\ &= \mathbf{e}^T(\lambda_1\mathbf{d}_1 + \lambda_2\mathbf{d}_2) \\ &= \lambda_1\mathbf{e}^T\mathbf{d}_1 + \lambda_2\mathbf{e}^T\mathbf{d}_2 \\ &= \lambda_1 + \lambda_2 \end{aligned}$$

- Hence, $\mathbf{d}^*$ can be written as $\mathbf{d}^* = \lambda_1\mathbf{d}_1 + (1-\lambda_1)\mathbf{d}_2$.
- As $\mathbf{d}_1$ is a direction of P, by a previous theorem $A\mathbf{d}_1 \leq \mathbf{0}$ and $\mathbf{d}_1 \geq \mathbf{0}$. Using also $\mathbf{e}^T\mathbf{d}_1 = 1$ we conclude $\mathbf{d}_1 \in P'$.
- Similarly we conclude $\mathbf{d}_2 \in P'$.
- Because $\mathbf{d}_1, \mathbf{d}_2 \in P'$ and $\mathbf{d}^* = \lambda_1\mathbf{d}_1 + (1-\lambda_1)\mathbf{d}_2$, it follows that $\mathbf{d}^*$ cannot be an extreme point of P' which contradicts our assumption.

□

Properties of extreme directions

Theorem

The collection of normalized extreme directions of P is finite.

Proof.

- By a previous theorem the normalized extreme directions of P are the extreme points of some associated polyhedron (P').
- By a previous theorem the number of extreme points of a polyhedron are finite.

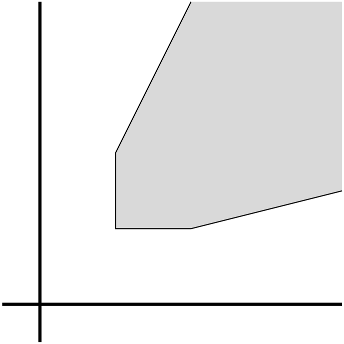
□

(The previous two theorems deal with polyhedra in the nonnegative orthant. The book discusses the general case.)

Pointed

Definition

If for a polyhedron P, it holds that $\text{lin}(P) = \{\mathbf{0}\}$, we call P a **pointed** polyhedron.



Example:

The polyhedron $P = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ is pointed.

Pointed

Theorem

Let P be a pointed polyhedron, the recession cone of P has

- *exactly one extreme point, namely* $\mathbf{0}$,
- *a finite number of extreme rays.*

The Minkowski-Weyl theorem

The Minkowski-Weyl theorem

Theorem (Minkowski-Weyl)

A set $P \subset \mathbb{R}^n$ is a polyhedron if and only if $P = Q + C$ for some polytope Q and some cone C with a finite number of extreme rays.

Corollary

Let P be a pointed polyhedron, with extreme points $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ and extreme directions $\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_p$. It follows that

$$
\begin{aligned}
P = \{\mathbf{x} \in \mathbb{R}^n | \mathbf{x} = \sum_{i=1}^{k} \lambda_i \mathbf{x}_i + \sum_{i=1}^{p} \mu_i \mathbf{d}_i \\
\sum_{i=1}^{k} \lambda_i = 1 \\
\lambda_i \geq 0 \qquad & \forall i = 1 \ldots, k \\
\mu_i \geq 0 \qquad & \forall i = 1 \ldots, p\}.
\end{aligned}
$$