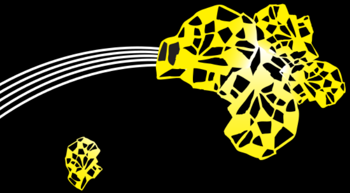
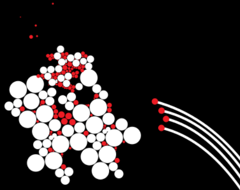
UNIVERSITY OF TWENTE.



Perfect Formulations
(Book Sections 4.1 – 4.4)



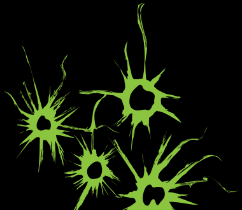
Matthias Walter

Topics:

- Integrality of polyhedra
- Totally unimodular matrices
- Application: bipartite matching / s-t-flows

Preknowledge:

- Polyhedra
- Cramer's rule
- Stable-set problem, matching problem, min-cost-flow problem



Agenda

1. Perfect Formulations
   - Integral Polyhedra
   - Total Unimodularity
   - A Criterion for Establishing Total Unimodularity
2. Application: Bipartite Matchings
   - Matchings
   - Incidence Matrices of Undirected Graphs
3. Application: Network Flows
   - Incidence Matrices of Directed Graphs / Network Flows
   - Maximum Flows & Minimum Cuts
   - Shortest Paths
4. Other Techniques to Establish Perfect Formulations
   - Laminar Set Families
   - Uncrossing
   - Intersections of Submodular Polytopes

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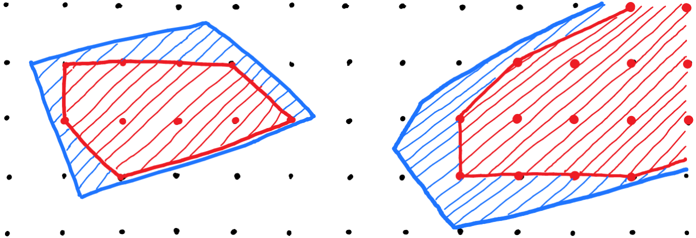
The Integer Hull and Integrality of a Polyhedron

Definitions – Integer hull and integrality

Let $P \subseteq \mathbb{R}^n$ be a polyhedron. The set $\operatorname{conv}(P \cap \mathbb{Z}^n)$ is called the **integer hull**. P is called **integral** if it is equal to its integer hull.



Definition – Perfect formulation

A MIP formulation with integer variables $I \subseteq [n]$ and LP relaxation P is called a **perfect formulation** if

$$\operatorname{conv}\{x \in P : x_i \in \mathbb{Z} \ \forall i \in I\} = P.$$

Remark:

- For IPs (i.e., $I = [n]$), a formulation with LP relaxation is P is perfect if and only if P is integral.

Total Unimodularity

Definition – Total unimodularity

A matrix $A \in \mathbb{R}^{m \times n}$ is **totally unimodular (TU)** if every square submatrix has determinant -1, 0 or $+1$.

Proposition – Properties of TU matrices

Total unimodularity is maintained under these operations:

1. Transposition
2. Permutation of rows or columns
3. Scaling rows or columns by -1.
4. Taking submatrices
5. Appending copies of rows or columns.
6. Appending unit rows or columns

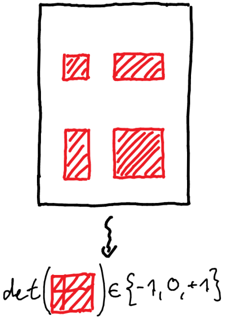
However:

- Total unimodularity is not maintained under appending other TU matrices:

$$A = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad [A \mid B] = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

- Elementary row/column operations may destroy TU: $\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & -1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 0 & -1 \end{pmatrix}$

Submatrices:



Properties of Total Unimodularity

Proposition – Properties of TU matrices

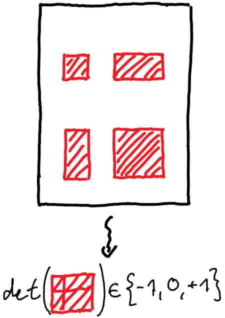
Total unimodularity is maintained under these operations:

1. Transposition
2. Permutation of rows or columns
3. Scaling rows or columns by -1.
4. Taking submatrices
5. Appending copies of rows or columns.
6. Appending unit rows or columns

Proof:

1. Transposition: for row subsets I and column subsets J we have $\det((A^\intercal)_{J,I}) = \det(A_{I,J})$.
2. Permutation of rows and columns: does not affect absolute value of determinant.
3. Scaling rows or columns by -1: does not affect absolute value of determinant.
4. Taking submatrices: by definition
5. Appending copies of rows or columns: if multiple copies participate in a submatrix, the determinant is 0.
6. Appending unit rows or columns: Apply Laplace rule for determinant calculation. ■

Reminder for TU:



TU Coefficient Matrix and Integral Right-hand-side imply Integrality of Polyhedron

Theorem – Implications of TU for polyhedra **[Hoffman & Kruskal, '56]**

Let $A \in \mathbb{R}^{m \times n}$ be TU and $b \in \mathbb{Z}^m$. Then $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ is integral.

Lemma – Cramer's Rule **[Cramer, 1750]**

Let $B \in \mathbb{Z}^{n \times n}$ be invertible. Then the unique solution to $Bx = d$ satisfies $x_i = \det(B^i) / \det(B)$ where B^i arises from B by replacing the i'th column with d.

Lemma 4.4 – Consequence of Cramer's Rule

Let $B \in \mathbb{Z}^{n \times n}$ and $d \in \mathbb{Z}^n$ be such that $|\det(B)| = 1$ holds. Then the unique solution to $Bx = d$ is integral.

Proof of the lemma:

- By Cramer's Rule, the unique solution is $x_i = \det(B^i) / \det(B)$.
- Since all entries of B^i are integer, also $\det(B^i)$ is an integer.
- Since the denominator is either -1 or $+1$, each x_i is integer. ■

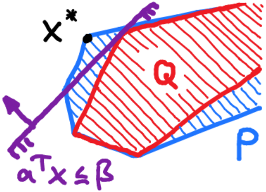
TU Coefficient Matrix and Integral Right-hand-side imply Integrality of Polyhedron

Theorem 4.4 – Implications of TU for polyhedra [Hoffman & Kruskal, '56]

Let $A \in \mathbb{R}^{m \times n}$ be TU and $b \in \mathbb{Z}^m$. Then $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ is integral.

Lemma – Consequence of Cramer's Rule

Let $B \in \mathbb{Z}^{n \times n}$ and $d \in \mathbb{Z}^n$ be such that $|\det(B)| = 1$ holds. Then the unique solution to $Bx = d$ is integral.



Proof:

- Let $Q := \text{conv}(P \cap \mathbb{Z}^n) \subseteq P$ be P's integer hull.
- Assuming $P \not\subseteq Q$, there must be an inequality $a^\intercal x \leq \beta$ that is valid for Q but not for P, i.e., $\max\{a^\intercal x : x \in P\} > \beta \geq \max\{a^\intercal x : x \in Q\}$.
- We can assume that the first LP is bounded: otherwise, add $-M \leq x_i \leq M$ for all $i \in [n]$ for sufficiently large M, which does not destroy TU by property (6).
- Let $x^\star \in \mathbb{R}^n$ be an optimal basic solution of the first LP. Note: $x^\star \notin Q$.
- There exists a subsystem $Bx \leq d$ of $Ax \leq b$ consisting of n inequalities such that $x^\star$ is the unique solution of $Bx = d$.
- The lemma implies $x^\star \in \mathbb{Z}^n$, and thus $x^\star \in Q$, a contradiction. ■

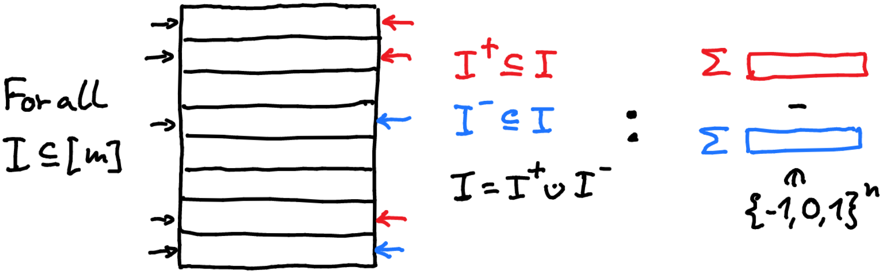
A Criterion for Establishing Total Unimodularity

Theorem 4.6 – Criterion of Ghoulia-Houri (row version) [Ghouila-Houri, '62]

A matrix $A \in \mathbb{R}^{m \times n}$ is TU if and only if **each subset** $I \subseteq [m]$ of rows can be partitioned into I^+ and I^- such that the following holds:

$$\sum_{i \in I^+} A_{i,\star} - \sum_{i \in I^-} A_{i,\star} \in \{-1, 0, +1\}^n. \qquad (1)$$

Proof: not in this lesson.



Hint: when applying it, we have to consider any $I \subseteq [m]$ and construct I^+ and I^-.

Reminder:

Matrix A is **TU** if $\det(B) \in \{-1, 0, +1\}$ holds for every square submatrix B.

Software for testing:

Combinatorial **M**atrix **R**ecognition

`discopt.github.io/cmr/`

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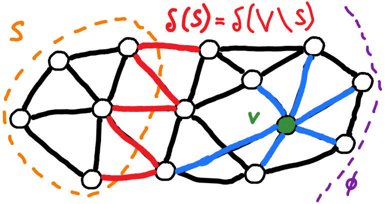
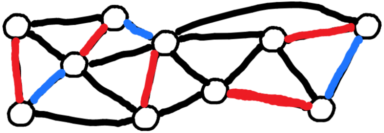
Cuts & Matchings

Definition – Cuts and shores in undirected graphs

Let $G = (V, E)$ be an undirected graph and $S \subseteq V$ be a node set. The edge set $\delta(S) := \{e \in E : |e \cap S| = 1\}$ is called the **cut induced by** S and S and $V \setminus S$ are called its **shores**. For $v \in V$ we write $\delta(v) := \delta(\{v\})$ for the **star cut**. The cut $\delta(\emptyset) = \delta(V) = \emptyset$ is called **trivial cut**.





Definition – Matching, perfect matching

Let $G = (V, E)$ be an undirected graph. An edge subset $M \subseteq E$ is called a **matching** of G if $|M \cap \delta(v)| \leq 1$ for every node $v \in V$. A matching M with $|M| = \frac{1}{2}|V|$ is called **perfect**.

The Matching Problem

Problem – Matching problem

- **Input:** Graph $G = (V, E)$ and weights $w \in \mathbb{R}^E$.
- **Feasible solutions:** Matchings $M \subseteq E$.
- **Goal:** Maximize $w(M) := \sum_{e \in M} w_e$.

Variables:

- $x_e \in \{0, 1\}$ for $e \in E$: $x_e = 1 \iff e$ belongs to the matching.

IP:

$$\max \sum_{e \in E} w_e x_e \tag{2a}$$

$$\text{s.t.} \sum_{e \in \delta(v)} x_e \leq 1 \qquad \forall v \in V \tag{2b}$$

$$x \in \{0, 1\}^E \tag{2c}$$

Two alternatives for perfect matchings:

$$\sum_{e \in E} x_e = \tfrac{1}{2} |V| \tag{3}$$

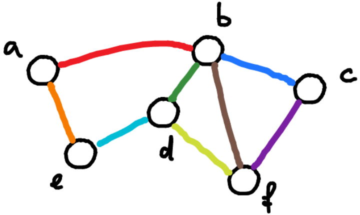
or

$$\sum_{e \in \delta(v)} x_e = 1 \qquad \forall v \in V \tag{4}$$

Incidence Matrices of Undirected Graphs

Definition – Incidence matrix of a graph

Let $G = (V, E)$ be a graph. Its **node-edge incidence matrix** is the matrix $M \in \{0,1\}^{V \times E}$ with $M_{v,e} = 1 \iff v \in e$.



$$\begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

IP formulation for matching:

$$\begin{aligned} \max \quad & \sum_{e \in E} w_e x_e && \text{(2a)} \\ \text{s.t.} \quad & Mx \leq 1 && \text{(2b)} \\ & x \in \{0,1\}^E && \text{(2c)} \end{aligned}$$

IP formulation for stable set:

$$\begin{aligned} \max \quad & \sum_{v \in V} w_v x_v && \text{(5a)} \\ \text{s.t.} \quad & M^\intercal x \leq 1 && \text{(5b)} \\ & x \in \{0,1\}^V && \text{(5c)} \end{aligned}$$

Incidence Matrices of Undirected Graphs

Theorem 4.18 – Total unimodularity of incidence matrix of a graph

Let $G = (V, E)$ be a graph. Its node-edge incidence matrix $M \in \{0,1\}^{V \times E}$ is totally unimodular if and only if G is bipartite.

Sufficiency proof:

- Let $G = (V, E)$ be a bipartite graph with bipartition $V = A \cup B$ and $M \in \{0, +1\}^{V \times E}$ be its node-edge incidence matrix.
- Let $I \subseteq V$ be a subset of M's rows. Each column of $M_{I,\star}$ has at most two 1's.
- Partitioning I into $I^+ := I \cap A$ and $I^- := I \cap B$ satisfies (1) since the two 1's in each column are not both in I^+ and not both in I^-.
- The result follows by the criterion of Ghouila-Houri.

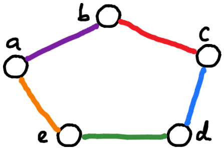
Necessity proof:

- Consider a cycle of odd length.
- Its incidence matrix has determinant ±2. ∎

Corollary – Perfect formulations for matching and stable-set

Let $G = (V, E)$ be a bipartite graph. Then IP formulations (2) and (5) are perfect formulations for the matching and stable-set problems, respectively.

Matrix for odd cycle:



$$\begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

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Incidence Matrices of Directed Graphs

Definition – Incidence matrix of a directed graph

Let $D = (V, A)$ be a digraph. Its **node-arc incidence matrix** is the matrix $M \in \{-1, 0, 1\}^{V \times A}$ defined via

$$M_{w,(u,v)} = \begin{cases} -1 & \text{if } w = u, \\ +1 & \text{if } w = v, \\ 0 & \text{otherwise.} \end{cases}$$

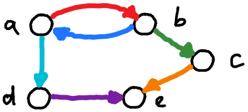
Theorem 4.9 – Total unimodularity of incidence matrix of a digraph

The node-arc incidence matrix of any digraph is totally unimodular.

Proof:

- Let $D = (V, A)$ be a digraph and $M \in \{-1, 0, +1\}^{V \times A}$ be its incidence matrix.
- Let $I \subseteq V$ be a subset of M's rows.
- Partitioning I into $I^+ := I$ and $I^- := \varnothing$ satisfies (1).
- The result follows by the criterion of Ghouila-Houri. ■

Example:



$$\begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} \begin{pmatrix} -1 & +1 & & & & -1 \\ +1 & -1 & -1 & & & \\ & & +1 & -1 & & \\ & & & & -1 & +1 \\ & & & +1 & +1 & \end{pmatrix}$$

Network Flows

Definition – Directed cuts

Let $D = (V, A)$ be a digraph and $S \subseteq V$ be a node set. The arc set $\delta^{\text{out}}(S) := \{(u, v) \in A : u \in S,\ v \notin S\}$ is called the **outgoing cut induced by** S. The set $\delta^{\text{in}}(S) := \delta^{\text{out}}(V \setminus S)$ is called the **incoming cut induced by** S. For $v \in V$ we write $\delta^{\text{out}}(v) := \delta^{\text{out}}(\{v\})$ and $\delta^{\text{in}}(v) := \delta^{\text{in}}(\{v\})$.

Definition – s-t-flow and flow polytope

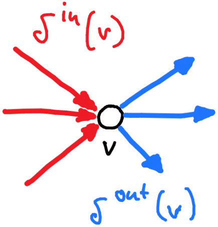
Let $D = (V, A)$ be a digraph with **source** and **sink** nodes $s, t \in V$, and let $u \in \mathbb{R}^A_{\geq 0}$ be **arc capacities**.

- An **s-t-flow** is a vector $f \in \mathbb{R}^A$ that satisfies (6).
- The set of all s-t-flows is called the **s-t-flow polytope** of (D, u).

Flow constraints:

$$\sum_{a \in \delta^{\text{in}}(v)} f_a - \sum_{a \in \delta^{\text{out}}(v)} f_a = 0 \quad \forall v \in V \setminus \{s, t\}, \tag{6a}$$

$$0 \leq f_a \leq u_a \quad \forall a \in A. \tag{6b}$$



Problem – Maximum s-t-flow problem

- **Input:** Digraph $D = (V, A)$, nodes $s, t \in V$, arc capacities $u \in \mathbb{R}^A_{\geq 0}$.
- **Feasible solutions:** s-t-flows $f \in \mathbb{R}^A$.
- **Goal:** Maximize **flow value** $\sum_{a \in \delta^{\text{in}}(t)} f_a - \sum_{a \in \delta^{\text{out}}(t)} f_a$.

Integrality of Flow Polytopes

Proposition – Constraint matrix of flow formulation

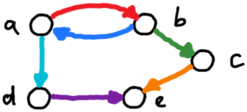
The constraint matrix for equations (6a) of the s-t-flow polytope is a submatrix of the node-arc incidence matrix of the digraph (obtained by removing the rows s, t).

Consequence of total unimodularity of node-arc incidence matrices:

Corollary – Integrality of flow polytopes

Let $D = (V, A)$ be a digraph with two nodes $s, t \in V$, and let $u \in \mathbb{Z}^A_{\geq 0}$ be **integral** arc capacities. Then the s-t-flow polytope is integral.

Example:



$$\begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} \begin{pmatrix} -1 & +1 & & & & -1 \\ +1 & -1 & -1 & & & \\ & & +1 & -1 & & \\ & & & & -1 & +1 \\ & & & +1 & +1 & \end{pmatrix}$$

Maximum Flows & Minimum Cuts

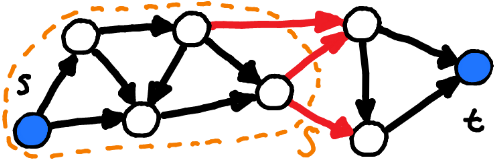
Definition – s-t-cut

Let $D = (V, A)$ be a digraph with nodes $s, t \in V$. An ***s*-*t*-cut** is a cut $\delta^{\text{out}}(S)$ induced by a set $S \subseteq V$ with $s \in S$ and $t \notin S$.

Problem – Minimum s-t-cut problem

- **Input:** Digraph $D = (V, A)$, source $s \in V$, sink $t \in V$, and arc capacities $u \in \mathbb{R}^A_{\geq 0}$.
- **Feasible solutions:** s-t-cuts $\delta^{\text{out}}(S)$.
- **Goal:** Minimize the **capacity** $\sum_{a \in \delta^{\text{out}}(S)} u_a$ of the cut.

An s-t-cut:



Theorem 4.15 – Max-Flow Min-Cut Theorem **[Ford & Fulkerson, '62]**

Let $D = (V, A)$ be a digraph with source $s \in V$, sink $t \in V$ and capacities $u \in \mathbb{R}^A_{\geq 0}$. Then the maximum value of an s-t-flow is equal to the minimum capacity of an s-t-cut.

Minimum Cost Flows

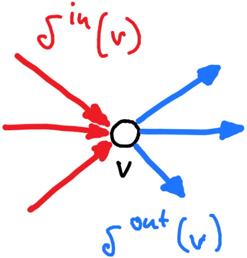
Definition – b-flows

Let $D = (V, A)$ be a digraph, $u \in \mathbb{R}^A_{\geq 0}$ be arc capacities and let $b \in \mathbb{R}^V$ be a **demand vector** that satisfies $\sum_{v \in V} b_v = 0$. A **b-flow** is a vector $f \in \mathbb{R}^A$ that satisfies (7).
A b-flow for $b = \mathbb{O}_V$ is called a **circulation**.

$$\sum_{a \in \delta^{\text{in}}(v)} f_a - \sum_{a \in \delta^{\text{out}}(v)} f_a = b_v \quad \forall v \in V, \tag{7a}$$

$$0 \leq f_a \leq u_a \quad \forall a \in A. \tag{7b}$$

Flow conservation:



Relation to maximum flow problem:

- Find largest $b_t = -b_s$ such that feasible b-flow with $b_v = 0$ for all $v \neq s, t$ exists.

Problem – Minimum cost b-flow/circulation problem

- **Input:** Digraph $D = (V, A)$, arc capacities $u \in \mathbb{R}^A_{\geq 0}$, costs $c \in \mathbb{R}^A$ and demands $b \in \mathbb{R}^V$ (circulations: $b = \mathbb{O}$).
- **Feasible solutions:** b-flows $f \in \mathbb{R}^A$.
- **Goal:** Minimize costs $\sum_{a \in A} c_a f_a$.

Shortest Paths via *b*-Flows

Problem – Shortest path problem

- **Input:** Digraph $D = (V, A)$, source $s \in V$, sink $t \in V$ and arc lengths $\ell \in \mathbb{R}^A$ that are **conservative**: $\ell(C) := \sum_{a \in C} \ell_a \geq 0$ for every cycle C in D.
- **Feasible solutions:** s-t-paths $P \subseteq A$.
- **Goal:** Minimize the length $\ell(P)$.

Variables:

- $f_a \in \{0,1\}$ for $a \in A$: $f_a = 1 \iff a$ is part of the path or a redundant cycle.

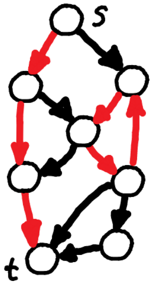
IP:

$$\min \sum_{a \in A} \ell_a f_a \tag{8a}$$

$$\text{s.t.} \quad \sum_{a \in \delta^{\text{in}}(v)} f_a - \sum_{a \in \delta^{\text{out}}(v)} f_a = \begin{cases} -1 & \text{if } v = s \\ +1 & \text{if } v = t \\ 0 & \text{otherwise} \end{cases} \qquad \forall v \in V \tag{8b}$$

$$f \in \{0,1\}^A \tag{8c}$$

A feasible solution:



Proposition – Correctness of shortest path formulation

The shortest path problem is correctly modeled by (8).

Correctness Proof for Flow Formulation for Shortest Paths

Proposition – Correctness of shortest path formulation

The shortest path problem is correctly modeled by (8).

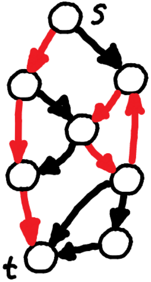
IP:

$$\min \sum_{a \in A} \ell_a f_a \tag{8a}$$

$$\text{s.t.} \quad \sum_{a \in \delta^{\text{in}}(v)} f_a - \sum_{a \in \delta^{\text{out}}(v)} f_a = \begin{cases} -1 & \text{if } v = s \\ +1 & \text{if } v = t \\ 0 & \text{otherwise} \end{cases} \qquad \forall v \in V \tag{8b}$$

$$f \in \{0,1\}^A \tag{8c}$$

A feasible solution:



Proof:

- Let $b \in \mathbb{R}^V$ be the right-hand side vector of (8b).
- For each path $P \subseteq A$, $\chi(P)$ is a b-flow with $\ell^\intercal \chi(P) = \ell(P)$.
- Let $f \in \{0,1\}^A$ be an ℓ-minimum (integral) b-flow f.
- By flow conservation, f contains an s-t-path.
- By ℓ-minimality and due to $\ell(C) \geq 0$ for each cycle C, we have that $f = \chi(P) + \chi(C_1) + \cdots \chi(C_k)$, where P is an ℓ-shortest s-t-path and C_i are cycles in D with $\ell(C_i) = 0$ for $i = 1, 2, \dots, k$.
- Remove cycles to extract P from f. Observe $\ell(P) = \ell^\intercal f$. ■

Perfect Formulation for Shortest Paths

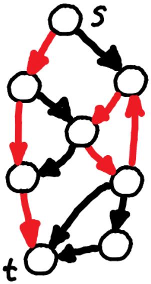
LP relaxation:

$$\min \sum_{a \in A} \ell_a f_a \tag{8a}$$

$$\text{s.t.} \sum_{a \in \delta^{\text{in}}(v)} f_a - \sum_{a \in \delta^{\text{out}}(v)} f_a = \begin{cases} -1 & \text{if } v = s \\ +1 & \text{if } v = t \\ 0 & \text{otherwise} \end{cases} \qquad \forall v \in V \tag{8b}$$

$$f \in \mathbb{R}^A_{\geq 0} \tag{8c'}$$

A feasible solution:



Consequence of total unimodularity of node-arc incidence matrices:

Corollary – Perfect shortest-path formulation

Formulation (8) is a perfect formulation for the shortest path problem.

Agenda

1. Perfect Formulations
   - Integral Polyhedra
   - Total Unimodularity
   - A Criterion for Establishing Total Unimodularity

2. Application: Bipartite Matchings
   - Matchings
   - Incidence Matrices of Undirected Graphs

3. Application: Network Flows
   - Incidence Matrices of Directed Graphs / Network Flows
   - Maximum Flows & Minimum Cuts
   - Shortest Paths

4. Other Techniques to Establish Perfect Formulations
   - Laminar Set Families
   - Uncrossing
   - Intersections of Submodular Polytopes

Laminar Set Families

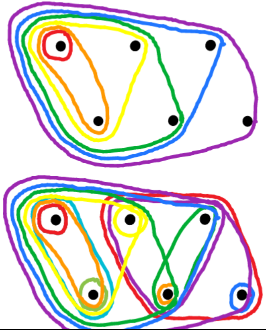
Definition – Laminar set family and incidence matrices

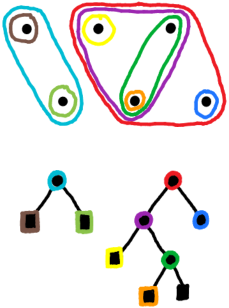
Let E be finite and let $\mathcal{L} \subseteq 2^E$ be a family of subsets. The **incidence matrix** of $\mathcal{L}$ is the matrix $M \in \{0,1\}^{\mathcal{L} \times E}$ defined via $M_{A,e} = 1 \iff e \in A$. We call $\mathcal{L}$ **laminar** if every two elements $A, B \in \mathcal{L}$ satisfy $A \subseteq B$ or $B \subseteq A$ or $A \cap B = \varnothing$.

Lemma – Incidence matrices of two laminar families **[Edmonds, '70]**

Let $\mathcal{L}$ be the union of two laminar families. Then its incidence matrix is TU.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$





$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Submodular Functions

Definitions – Submodular, monotone and normalized set function

Let E be a finite ground set. A function $f : 2^E \to \mathbb{R}$ is called

❶ **submodular** if $f(S \cap T) + f(S \cup T) \leq f(S) + f(T)$ holds for all $S, T \subseteq E$,

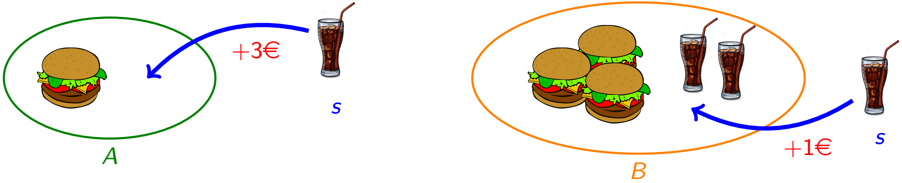
❷ **monotone** if $f(S) \leq f(T)$ holds for all $S \subseteq T \subseteq E$, and

❸ **normalized** if $f(\varnothing) = 0$.

Lemma – Diminishing returns

f is submodular if and only if for all $A \subseteq B \subseteq E$ and each $s \in E \setminus B$, we have

$$f(A \cup \{s\}) - f(A) \geq f(B \cup \{s\}) - f(B). \tag{9}$$



Submodular Functions and Uncrossing

Lemma (Exercise 4.25) – Uncrossing for Submodular Functions

Let $f : 2^E \to \mathbb{R}$ be a submodular normalized function. Let $\bar{x}$ be a vertex of the polyhedron $P = \{x \in \mathbb{R}^E : \sum_{e \in S} x_e \leq f(S) \text{ for all } S \subseteq E\}$. Then $\bar{x}$ satisfies at equality $|E|$ linearly independent inequalities $\sum_{e \in S} x_e = f(S_i)$ for $i = 1, 2, \dots, |E|$ such that the family $\mathcal{L} := \{S_i \mid i = 1, 2, \dots, |E|\}$ is laminar.

Proof:

- Consider among all such families $\mathcal{L}$ one that maximizes $\varphi(\mathcal{L}) := \sum_{S \in \mathcal{L}} |S|^2$.
- Suppose there exist sets $S, T \in \mathcal{L}$ that cross.
- Since $\bar{x}$ satisfies the two inequalities with equality and since f is submodular, we obtain

$$f(S) + f(T) = \sum_{e \in S} \bar{x}_e + \sum_{e \in T} \bar{x}_e = \sum_{e \in S \cap T} \bar{x}_e + \sum_{e \in S \cup T} \bar{x}_e \leq f(S \cap T) + f(S \cup T) \leq f(S) + f(T)$$

- Thus, equality holds throughout.
- Hence, also the inequalities for $S \cap T$ and $S \cup T$ are satisfied with equality.
- We can replace S and T by $S \cap T$ and $S \cup T$ since both coefficient vectors pairs span the same space.
- This would increase $\varphi(\mathcal{L})$ due to $|S \cup T|^2 + |S \cap T|^2 - |S|^2 - |T|^2 = 2 \cdot |S \setminus T| \cdot |T \setminus S| > 0$, a contradiction to the choice of $\mathcal{L}$. ■

Intersections of Submodular Polytopes

Definition – Submodular polytope

Let E be a finite ground set and let $f : 2^E \to \mathbb{R}$ be submodular, monotone and normalized. Its **submodular polytope** is defined as

$$P_{\text{sub}}^{(f)} := \{x \in \mathbb{R}^E_{\geq 0} : \sum_{e \in S} x_e \leq f(S) \quad \forall S \subseteq E\}.$$

Theorem – Intersection of two submodular polytopes is integral [Edmonds, '70]

Let $f_1, f_2 : 2^E \to \mathbb{R}$ be integer-valued, submodular, monotone and normalized. Then $P_{\text{sub}}^{(f_1)} \cap P_{\text{sub}}^{(f_2)}$ is an integral polytope.

Proof:

- Combine lemmas about laminar families and uncrossing. ■

Main combinatorial example:

- Rank functions of matroids are submodular.

Examples of matroids:

- Linearly independent subsets of a finite set of vectors.
- Sets of at most a certain cardinality.
- Forests in a graph.
- Node sets that can be can be covered by a matching.

Lesson Recap – **Any Questions?**

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