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Extended Formulations (Book Section 4.9)

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Topics:

- Extended Formulations
- Extended Formulations via Unions Polyhedra
- Extended Formulations via Flows

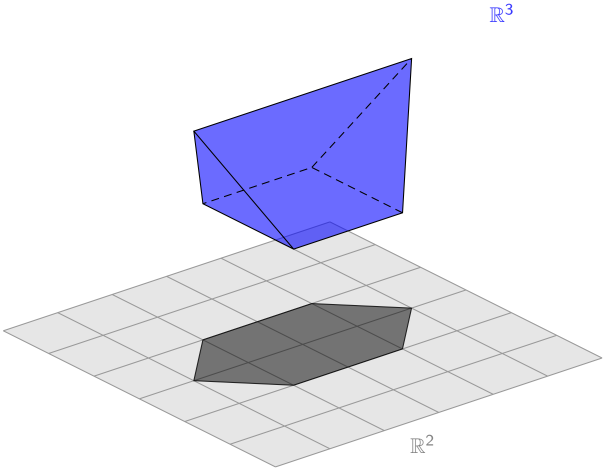
Preknowledge:

- Polyhedra
- Network Flow Formulations

Agenda

Agenda

Geometry of Extended Formulations



Extended Formulations

Definition (from Lecture 4) – Orthogonal Projection

The **orthogonal projection** of a set $Q \subseteq \mathbb{R}^{n+p}$ onto $\mathbb{R}^n$ is

$$\text{proj}_x(Q) := \{x \in \mathbb{R}^n \mid \exists z \in \mathbb{R}^p,\ (x, z) \in Q\}$$

Definition (from Lecture 4) – Extended Formulation and Size

A set of linear inequalities describing Q is called an extended formulation of $P := \text{proj}_x(Q)$. The **size** of the extended formulation is the number m of inequalities.

Remarks:

- The concept does not change if we allow affine/linear projections: if $x = Tz$, we can add these equations to the extended formulation and orthogonally project onto x.
- Linear optimization over P can be reduced to linear optimization over Q.
- The size ignores the number of variables and equations as these can be reduced to be in $\mathcal{O}(n + p)$:
  1. While Q is unbounded in some direction that projects to $\mathbb{O}$, we can slice it (= add an equation) without changing the projection image.
  2. Then we can project out variables in order to remove equations.
  3. Finally, a full-dimensional pointed polyhedron in dimension p has at least p inequalities, so p cannot be too large.

Agenda

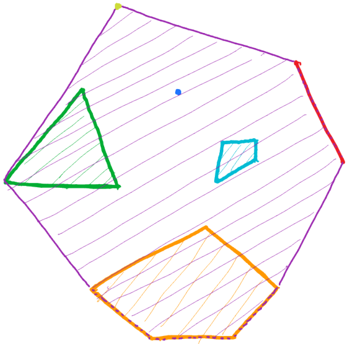
Union of Polyhedra

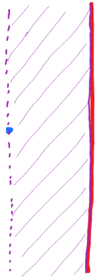
Goal:

- For polyhedra $P_1, P_2, \dots, P_k$, we want to describe the convex hull of their union

$$P := \mathrm{conv}(P_1 \cup P_2 \cup \dots \cup P_k).$$

Geometry:





Observation: P may not be a polyhedron.

This course: We only address the case in which $P_1, P_2, \dots, P_k$ are bounded.

Book: Technicalities for unbounded case are in Section 4.9 of the book.

Extended Formulation for Union of Polytopes

Consider k **polytopes** $P_1, P_2, \ldots, P_k \subseteq \mathbb{R}^n$ defined via $P_i := \{x \in \mathbb{R}^n : A_i x \leq b_i\}$ and denote by $P = \text{conv}(P_1 \cup P_2 \cup \cdots \cup P_k)$ the convex hull of their union.

LP:

$$\sum_{i=1}^{k} y_i = y \tag{1a}$$

$$A_i y_i \leq b_i x_i \quad \text{for } i = 1, 2, \ldots, k \tag{1b}$$

$$\sum_{i=1}^{k} x_i = 1 \tag{1c}$$

$$x_i \in \mathbb{R}_{\geq 0} \text{ and } y_i \in \mathbb{R}^n \quad \text{for } i = 1, 2, \ldots, k \tag{1d}$$

$$y \in \mathbb{R}^n \tag{1e}$$

Variables:

- $x_i = 1 \Rightarrow$ target point y lies in polytope P_i.
- If $x_i = 1$ then $y_i = y$, and if $x_i = 0$ then $y_i = \mathbb{O}$.

Theorem 4.39 – Extended formulation for union of polytopes [Balas 1974]

Formulation (1) with the projection on y is a **perfect** extended formulation for P. Its size is k plus the sum of the number of inequalities of the P_i.

Theorem 4.39 – Extended formulation for union of polytopes [Balas 1974]

Formulation (1) with the projection on y is a **perfect** extended formulation for P. Its size is k plus the sum of the number of inequalities of the P_i.

Proof:

- Let $\bar{z} = (\bar{y}, \bar{y}_1, \dots, \bar{y}_k, \bar{x}_1, \dots, \bar{x}_2)$ be in the polyhedron defined by (1).
- We have to show that $\bar{y}$ lies in P.
- For t such that $\bar{x}_t > 0$, define the point $z^t = (y^t, y^t_1, \dots, y^t_k, x^t_1, \dots, x^t_k)$ with

$$y^t := \frac{\bar{y}^t}{\bar{x}^t}, \qquad y^t_i := \begin{cases} \frac{\bar{y}_i}{\bar{x}_i} & \text{for } i = t \\ 0 & \text{otherwise} \end{cases}, \qquad x^t_i := \begin{cases} 1 & \text{for } i = t \\ 0 & \text{otherwise} \end{cases}$$

- Points z^t are feasible for (1) with $y^t_i = \mathbb{O}$ for $i \neq t$ and $y^t_t \in P_t$.
- We verify that $\bar{z}$ is a convex combination of these: $\bar{z} = \sum_{t:\bar{x}_t \neq 0} \bar{x}_t z^t$
- Let $y \in P$ and fix a convex combination $y = \sum_{i=1}^k \lambda_k y_i$ with $y_i \in P_i$, $\sum_{i=1}^k \lambda_i = 1$ and $\lambda_i \geq 0$ for $i = 1, 2, \dots, k$.
- For $i \in [k]$, define $(\bar{y}_i, \bar{x}_i) := (\mathbb{O}, 0)$ if $\lambda_i = 0$ and $(\bar{y}_i, \bar{x}_i) := (\lambda_i y_i, \lambda_i)$ otherwise.
- The constraints of (1) are easily checked. ∎

Reminder:

$$\sum_{i=1}^k y_i = y \qquad (1a)$$

$$A_i y_i \leq b_i x_i \qquad (1b)$$

$$\sum_{i=1}^k x_i = 1 \qquad (1c)$$

$$x \geq \mathbb{O} \qquad (1d)$$

Union of Polytopes and Extended Formulations

What if the P_i are given via extended formulations?

- $P_i = \text{proj}_y(\{(y,z) \in \mathbb{R}^{n+p_i} : A_i y + C_i z \leq b_i\})$.

Modification of (1):

- $A_i y_i \leq b_i x_i$ is replaced by $A_i y_i + C_i z \leq b_i x_i$.

Corollary – Union of extended formulations

Let $P_1, P_2, \ldots, P_k \subseteq \mathbb{R}^n$ be polytopes. Then the convex hull $\text{conv}(P_1 \cup P_2 \cup \cdots \cup P_k)$ of their union has an extended formulation of size k plus the sum of the (minimum) sizes of extended formulations of the P_i.

Remark:

- Additional $+k$ are due to $x_i \geq 0$ for each $i \in \{1, 2, \ldots, k\}$. This $+1$ can be skipped if P_i is a polytope with $\dim(P_i) \geq 1$.

Reminder:

$$\sum_{i=1}^k y_i = y \tag{1a}$$

$$A_i y_i \leq b_i x_i \tag{1b}$$

$$\sum_{i=1}^k x_i = 1 \tag{1c}$$

$$x \geq \mathbb{O} \tag{1d}$$

Even Parity Polytope

Task:

- Describe the (convex hull P^n_{even} of the) set of $x \in \{0,1\}^n$ with $\sum_{i=1}^n x_i$ even.
- Arises as a substructure in many problems: cyclic routes cross every cut in a graph an even number of times.
- Optimization is easy: solve over $[0,1]^n$ and potentially flip cheapest coordinate.

Theorem – Perfect formulation for even parity polytope **[Jeroslow, 1975]**

P^n_{even} is described by these (for $n \geq 3$ facet-defining) inequalities.

$$\sum_{i \in N \setminus S} x_i + \sum_{i \in S} (1 - x_i) \geq 1 \qquad \text{for all } S \subseteq \{1,2,\ldots,n\} \text{ with } |S| \text{ odd} \qquad (2a)$$

$$x_i \in [0,1] \qquad \text{for } i = 1,2,\ldots,n \qquad (2b)$$

Theorem – Disjunctive program for even parity polytope

P^n_{even} has an extended formulation of size $\mathcal{O}(n^2)$.

Proof:

- For $k \in \mathbb{Z}$, $P_k := \left\{x \in [0,1]^n \mid \sum_{i=1}^n x_i = k\right\}$ is integral.
- $P^n_{\text{even}} = \text{conv}(P_0 \cup P_2 \cup \cdots \cup P_{2 \cdot \lfloor n/2 \rfloor})$.

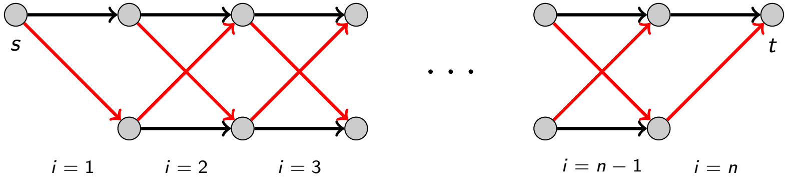
Agenda

Even Parity Polytope via Flows

Theorem – Flow Extended Formulation [Carr & Konjevod 2004]

P^n_{even} has an extended formulation of size $4n - 4$.

Construction:



- Let Q be the s-t-flow polytope of digraph $D = (V, A)$.
- The vertices of Q are incidence vectors of s-t-paths in D.
- Define $\pi : \mathbb{R}^A \to \mathbb{R}^n$ via

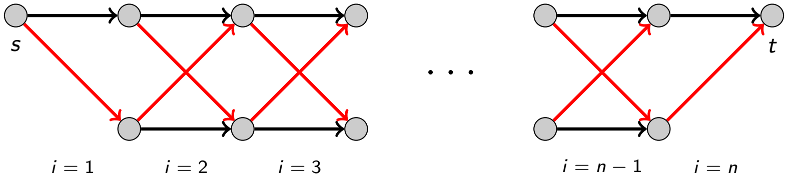
$$\pi(y)_i := \begin{cases} y_{a_i} & \text{for } i = 1 \\ y_{a_i} + y_{b_i} & \text{for } i = 2, 3, \dots, n-1 \\ y_{b_i} & \text{for } i = n. \end{cases}$$

- The formulation has one inequality per arc.

Claim – Projection

$\pi(Q) = P^n_{\text{even}}$ holds.

Extended Formulation for Even Subsets



Claim – Projection

$\pi(Q) = P^n_{\text{even}}$ holds.

Proof of the claim:

$$\begin{aligned}\pi(Q) &= \pi(\text{conv}\{\chi(W) : W \subseteq A \text{ is } s\text{-}t\text{-path in } D\}) \\ &= \text{conv}\{\pi(\chi(W)) : W \subseteq A \text{ is } s\text{-}t\text{-path in } D\} \\ &= \text{conv}\left\{v \in \{0,1\}^n : \sum_{i=1}^n v_i \in 2\mathbb{Z}\right\} \\ &= P^n_{\text{even}}\end{aligned}$$

Max-Flow Min-Cut Theorem

Definition – s-t-flow and flow polytope

Let $D = (V, A)$ be a digraph with **source** and **sink** nodes $s, t \in V$, and let $u \in \mathbb{R}^A_{\geq 0}$ be **arc capacities**.

- An **s-t-flow** is a vector $f \in \mathbb{R}^A$ that satisfies (3).
- The set of all s-t-flows is called the **s-t-flow polytope** of (D, u).

Flow constraints:

$$\sum_{a \in \delta^{\text{in}}(v)} f_a - \sum_{a \in \delta^{\text{out}}(v)} f_a = 0 \quad \forall v \in V \setminus \{s, t\}, \tag{3a}$$

$$0 \leq f_a \leq u_a \quad \forall a \in A. \tag{3b}$$

Notation:

$\delta^{\text{out}}(S)$ denotes all arcs $(u, v) \in A$ that have $u \in S$ and $v \notin S$.

$\delta^{\text{out}}(v) := \delta^{\text{out}}(\{v\})$.

Definition – s-t-cut

Let $D = (V, A)$ be a digraph with nodes $s, t \in V$. An *s*-*t*-**cut** is a cut $\delta^{\text{out}}(S)$ induced by a set $S \subseteq V$ with $s \in S$ and $t \notin S$.

Theorem – Max-Flow Min-Cut Theorem **[Ford & Fulkerson, '62]**

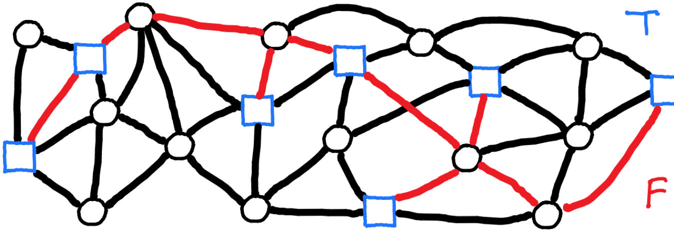
Let $D = (V, A)$ be a digraph with source $s \in V$, sink $t \in V$ and capacities $u \in \mathbb{R}^A_{\geq 0}$. Then the maximum value of an s-t-flow is equal to the minimum capacity of an s-t-cut.

The Steiner Tree Problem

Problem – Steiner tree problem

- **Input:** Graph $G = (V, E)$, **terminals** $T \subseteq V$, edge costs $c \in \mathbb{R}^E_{\geq 0}$.
- **Feasible solutions:** Subsets $F \subseteq E$, called **Steiner trees**, such that (V, F) contains an s-t-path for each pair of terminals $s, t \in T$.
- **Goal:** Minimize the cost of $c(F) := \sum_{e \in F} c_e$.

Example:



Applications:

Prototype problem for network design:

- telecommunication
- water/gas supply
- wiring of a chip
- ...

Remark:

- After removing edges of zero cost, an optimal Steiner tree is indeed a tree, i.e., it is connected and contains no cycles.

The Undirected Cut Formulation for Steiner Trees

Variables:

- $x_e \in \{0,1\}$ for $e \in E$: $x_e = 1 \iff e$ belongs to Steiner tree.

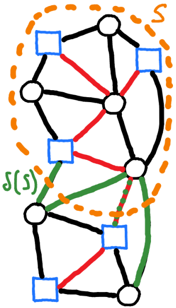
IP:

$$\min \sum_{e \in E} c_e x_e \tag{4a}$$

$$\text{s.t.} \sum_{e \in \delta(S)} x_e \geq 1 \qquad \forall S \subseteq V : S \cap T \neq \varnothing, T \setminus S \neq \varnothing \tag{4b}$$

$$x \in \{0,1\}^E \tag{4c}$$

A Steiner cut:



Proposition – Correctness of undirected cut formulation

Formulation (4) correctly models the Steiner tree problem.

Proof:

- Let F be a Steiner tree, $x := \chi(F)$ and let S be as in (4b).
- Choose $s \in T \cap S$ and $t \in T \setminus S$. There exists an s-t-path P in F.
- It satisfies $P \cap \delta(S) \neq \varnothing$, showing that (4b) is satisfied.

- Let x be feasible for (4). We have to show that $F := \text{supp}(x)$ connects T.
- Suppose there is a pair $s, t \in T$ that is not connected in F.
- Let $S \subseteq V$ be the set of nodes reachable from s in F. Note: $t \notin S$.
- By construction, $F \cap \delta(S) = \varnothing$, contradicting (4b). ■

Undirected Extended Flow Formulation for Steiner Trees

Auxiliary data:

- We fix a **root node** $r \in T$.
- Let $D = (V, A)$ with $A \coloneqq \{(u, v) : \{u, v\} \in E\}$ be the **bidirected graph** of G.

Variables:

- $x_e \in \{0, 1\}$ for $e \in E$: $x_e = 1 \iff e$ belongs to Steiner tree.
- $f_a^k \in \{0, 1\}$ for $a \in A$ and $k \in T \setminus \{r\}$: f^k models r-k-flow of value 1.

IP:

$$\min \sum_{e \in E} c_e x_e \tag{5a}$$

$$\text{s.t.} \sum_{a \in \delta^{\text{in}}(v)} f_a^k - \sum_{a \in \delta^{\text{out}}(v)} f_a^k = \begin{cases} -1 & \text{if } v = r \\ +1 & \text{if } v = k \\ 0 & \text{otherwise} \end{cases} \quad \forall v \in V,\ \forall k \in T \setminus \{r\} \tag{5b}$$

$$f_{(u,v)}^k \leq x_{\{u,v\}} \quad \forall (u, v) \in A,\ \forall k \in T \setminus \{r\} \tag{5c}$$

$$f^k \in \{0, 1\}^A \quad \forall k \in T \setminus \{r\} \tag{5d}$$

$$x \in \{0, 1\}^E \tag{5e}$$

Undirected Extended Flow Formulation for Steiner Trees

Theorem – Undirected extended flow formulation for Steiner trees

The LP relaxation of (5) is an extended formulation of the LP relaxation of (4) (via projection on the x-variables).

Proof strategy:

- Define $Q := \{(x,f) \in \mathbb{R}^E_{\geq 0} \times \mathbb{R}^* : (x,f)$ satisfies (5b) and (5c)$\}$ and $P := \{x \in \mathbb{R}^E_{\geq 0} : x$ satisfies (4b)$\}$.
- We show $P = \text{proj}_x(Q)$ by first showing $\text{proj}_x(Q) \subseteq P$ and then showing $P \subseteq \text{proj}_x(Q)$.
- For the first part, we consider some x which is a projection of some vector (x,f) satisfying (5b) and (5c). We will show that x satisfies (4b).
- For the second part, we consider some $x \in P$, satisfying (4b). We will show that there exists a vector f such that (x,f) satisfy (5b) and (5c).

IP (4):

(4b) $\sum_{e \in \delta(S)} x_e \geq 1$ for each $S \subseteq V$ with $S \cap T \neq \varnothing$ and $T \setminus S \neq \varnothing$

IP (5):

(5b) r-k-flows $f^k \in \mathbb{R}^A$ of value 1 for each $k \in T \setminus \{r\}$

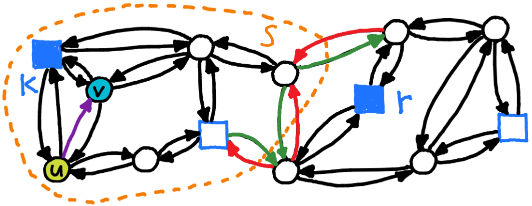
(5c) $f^k_{(u,v)} \leq x_{\{u,v\}}$

Cut Relaxation is Contained in Projection of Flow Relaxation

Proof part 1:

- Let (x, f) satisfy (5b) and (5c).
- We show that x satisfies (4b) for each $S \subseteq V$ with $r \notin S$ and $T \cap S \neq \varnothing$.
- Choose $k \in T \cap S$ and consider the sum of (5b) for this k and all $v \in S$:

$$\sum_{v \in S} \Big(\sum_{a \in \delta^{\text{in}}(v)} f_a^k - \sum_{a \in \delta^{\text{out}}(v)} f_a^k \Big) = 1 + \sum_{v \in S \setminus \{k\}} 0$$



- Observing that the flow on arcs inside S cancels out, we obtain

$$1 = \sum_{a \in \delta^{\text{in}}(S)} f_a^k - \sum_{a \in \delta^{\text{out}}(S)} f_a^k \leq \sum_{(u,v) \in \delta^{\text{in}}(S)} x_{\{u,v\}} - \sum_{a \in \delta^{\text{out}}(S)} 0 = \sum_{e \in \delta(S)} x_e,$$

which is (4b). ■

IP (4):

(4b) $\sum_{e \in \delta(S)} x_e \geq 1$ for each $S \subseteq V$ with $S \cap T \neq \varnothing$ and $T \setminus S \neq \varnothing$

IP (5):

(5b) r-k-flows $f^k \in \mathbb{R}^A$ of value 1 for each $k \in T \setminus \{r\}$

(5c) $f^k_{(u,v)} \leq x_{\{u,v\}}$

Projection of Flow Relaxation is Contained in Cut Relaxation

Proof part 2:

- Let x satisfy (4b).
- We have to show for each $k \in T \setminus \{r\}$ that there exists an r-k-flow $f^k \in \mathbb{R}^A_{\geq 0}$ with flow value 1 (i.e., f^k satisfies (5b)) that respects arc capacities $x_{\{u,v\}}$ on all arcs $(u,v) \in A$ (i.e., f^k satisfies (5c)).
- The inequality $\sum_{e \in \delta(S)} x_e \geq 1$ is satisfied for all $S \subseteq V$ with $r \in S$ and $k \notin S$.
- Hence, the capacity of the minimum r-k-cut is $\gamma \geq 1$.
- By the Max-Flow Min-Cut Theorem, there exists an r-k-flow of value γ.
- Scaling this flow by $\frac{1}{\gamma}$ yields a flow of value 1 that also respects the capacities (since we scale down). ∎

IP (4):

(4b) $\sum_{e \in \delta(S)} x_e \geq 1$ for each $S \subseteq V$ with $S \cap T \neq \varnothing$ and $T \setminus S \neq \varnothing$

IP (5):

(5b) r-k-flows $f^k \in \mathbb{R}^A$ of value 1 for each $k \in T \setminus \{r\}$

(5c) $f^k_{(u,v)} \leq x_{\{u,v\}}$

Lesson Recap – **Any Questions?**

1. Extended Formulations
   - Introduction

2. Extended Formulations via Union of Polyhedra
   - Union of Polyhedra
   - Application: Even Parity Polytope

3. Extended Formulations via Flows
   - Application: Even Parity Polytope
   - Application: Steiner Trees – Undirected Cut Formulation
   - Application: Steiner Trees – Undirected Flow Formulation