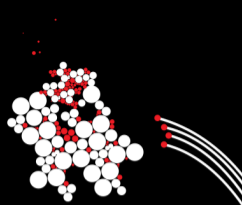


Matthias Walter

Extended Formulations (Book Section 4.9)

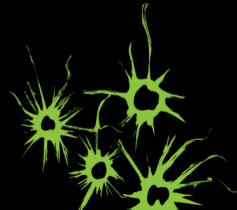


Topics:

- ▶ Extended Formulations
- ▶ Extended Formulations via Unions Polyhedra
- ▶ Extended Formulations via Flows

Preknowledge:

- ▶ Polyhedra
- ▶ Network Flow Formulations



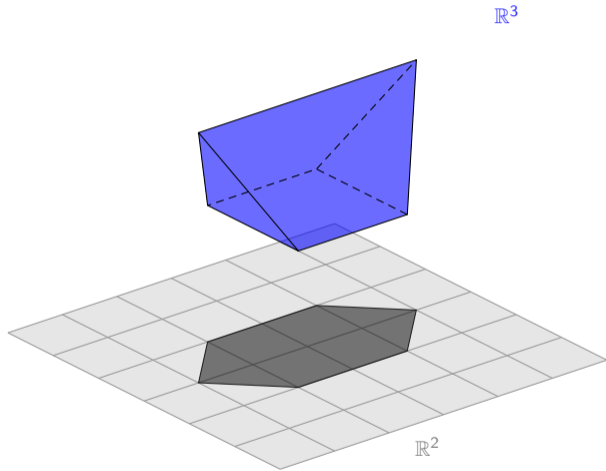
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- 1 Extended Formulations
 - Introduction
- 2 Extended Formulations via Union of Polyhedra
 - Union of Polyhedra
 - Application: Even Parity Polytope
- 3 Extended Formulations via Flows
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Geometry of Extended Formulations



Definition (from Lecture 4) – Orthogonal Projection

The **orthogonal projection** of a set $Q \subseteq \mathbb{R}^{n+p}$ onto \mathbb{R}^n is

$$\text{proj}_x(Q) := \{x \in \mathbb{R}^n \mid \exists z \in \mathbb{R}^p, (x, z) \in Q\}$$

Definition (from Lecture 4) – Extended Formulation and Size

A set of linear inequalities describing Q is called an extended formulation of $P := \text{proj}_x(Q)$. The **size** of the extended formulation is the number m of inequalities.

Remarks:

- ▶ The concept does not change if we allow affine/linear projections: if $x = Tz$, we can add these equations to the extended formulation and orthogonally project onto x .
- ▶ Linear optimization over P can be reduced to linear optimization over Q .
- ▶ The size ignores the number of variables and equations as these can be reduced to be in $\mathcal{O}(n+p)$:
 - ① While Q is unbounded in some direction that projects to $\mathbb{0}$, we can slice it (= add an equation) without changing the projection image.
 - ② Then we can project out variables in order to remove equations.
 - ③ Finally, a full-dimensional pointed polyhedron in dimension p has at least p inequalities, so p cannot be too large.

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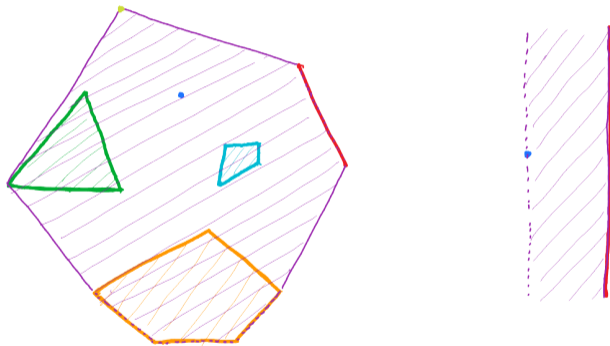
Union of Polyhedra

Goal:

- For polyhedra P_1, P_2, \dots, P_k , we want to describe the convex hull of their union

$$P := \text{conv}(P_1 \cup P_2 \cup \dots \cup P_k).$$

Geometry:



Observation: P may not be a polyhedron.

This course: We only address the case in which P_1, P_2, \dots, P_k are bounded.

Book: Technicalities for unbounded case are in Section 4.9 of the book.

Extended Formulation for Union of Polytopes

Consider k **polytopes** $P_1, P_2, \dots, P_k \subseteq \mathbb{R}^n$ defined via $P_i := \{x \in \mathbb{R}^n : A_i x \leq b_i\}$ and denote by $P = \text{conv}(P_1 \cup P_2 \cup \dots \cup P_k)$ the convex hull of their union.

LP:

$$\sum_{i=1}^k y_i = y \quad (1a)$$

$$A_i y_i \leq b_i x_i \quad \text{for } i = 1, 2, \dots, k \quad (1b)$$

$$\sum_{i=1}^k x_i = 1 \quad (1c)$$

$$x_i \in \mathbb{R}_{\geq 0} \text{ and } y_i \in \mathbb{R}^n \quad \text{for } i = 1, 2, \dots, k \quad (1d)$$

$$y \in \mathbb{R}^n \quad (1e)$$

Variables:

- ▶ $x_i = 1 \Rightarrow$ target point y lies in polytope P_i .
- ▶ If $x_i = 1$ then $y_i = y$, and if $x_i = 0$ then $y_i = \mathbb{0}$.

Theorem 4.39 – Extended formulation for union of polytopes [Balas 1974]

Formulation (1) with the projection on y is a **perfect** extended formulation for P . Its size is k plus the sum of the number of inequalities of the P_i .

Theorem 4.39 – Extended formulation for union of polytopes [Balas 1974]

Formulation (1) with the projection on y is a **perfect** extended formulation for P . Its size is k plus the sum of the number of inequalities of the P_i .

Proof:

- ▶ Let $\bar{z} = (\bar{y}, \bar{y}_1, \dots, \bar{y}_k, \bar{x}_1, \dots, \bar{x}_k)$ be in the polyhedron defined by (1).
- ▶ We have to show that \bar{y} lies in P .
- ▶ For t such that $\bar{x}_t > 0$, define the point $z^t = (y^t, y_1^t, \dots, y_k^t, x_1^t, \dots, x_k^t)$ with

$$y^t := \frac{\bar{y}^t}{\bar{x}^t}, \quad y_i^t := \begin{cases} \frac{\bar{y}_i}{\bar{x}_i} & \text{for } i = t \\ 0 & \text{otherwise} \end{cases}, \quad x_i^t := \begin{cases} 1 & \text{for } i = t \\ 0 & \text{otherwise} \end{cases}$$

- ▶ Points z^t are feasible for (1) with $y_i^t = 0$ for $i \neq t$ and $y_t^t \in P_t$.
- ▶ We verify that \bar{z} is a convex combination of these: $\bar{z} = \sum_{t: \bar{x}_t \neq 0} \bar{x}_t z^t$

- ▶ Let $y \in P$ and fix a convex combination $y = \sum_{i=1}^k \lambda_i y_i$ with $y_i \in P_i$, $\sum_{i=1}^k \lambda_i = 1$ and $\lambda_i \geq 0$ for $i = 1, 2, \dots, k$.
- ▶ For $i \in [k]$, define $(\bar{y}_i, \bar{x}_i) := (0, 0)$ if $\lambda_i = 0$ and $(\bar{y}_i, \bar{x}_i) := (\lambda_i y_i, \lambda_i)$ otherwise.
- ▶ The constraints of (1) are easily checked. ■

Reminder:

$$\sum_{i=1}^k y_i = y \quad (1a)$$

$$A_i y_i \leq b_i x_i \quad (1b)$$

$$\sum_{i=1}^k x_i = 1 \quad (1c)$$

$$x \geq 0 \quad (1d)$$

Union of Polytopes and Extended Formulations

What if the P_i are given via extended formulations?

- ▶ $P_i = \text{proj}_y(\{(y, z) \in \mathbb{R}^{n+p_i} : A_i y + C_i z \leq b_i\})$.

Modification of (1):

- ▶ $A_i y_i \leq b_i x_i$ is replaced by $A_i y_i + C_i z \leq b_i x_i$.

Corollary – Union of extended formulations

Let $P_1, P_2, \dots, P_k \subseteq \mathbb{R}^n$ be polytopes. Then the convex hull $\text{conv}(P_1 \cup P_2 \cup \dots \cup P_k)$ of their union has an extended formulation of size k plus the sum of the (minimum) sizes of extended formulations of the P_i .

Remark:

- ▶ Additional $+k$ are due to $x_i \geq 0$ for each $i \in \{1, 2, \dots, k\}$. This $+1$ can be skipped if P_i is a polytope with $\dim(P_i) \geq 1$.

Reminder:

$$\sum_{i=1}^k y_i = y \quad (1a)$$

$$A_i y_i \leq b_i x_i \quad (1b)$$

$$\sum_{i=1}^k x_i = 1 \quad (1c)$$

$$x \geq \mathbb{0} \quad (1d)$$

Even Parity Polytope

Task:

- ▶ Describe the (convex hull P_{even}^n of the) set of $x \in \{0, 1\}^n$ with $\sum_{i=1}^n x_i$ even.
- ▶ Arises as a substructure in many problems: cyclic routes cross every cut in a graph an even number of times.
- ▶ Optimization is easy: solve over $[0, 1]^n$ and potentially flip cheapest coordinate.

Theorem – Perfect formulation for even parity polytope

[Jeroslow, 1975]

P_{even}^n is described by these (for $n \geq 3$ facet-defining) inequalities.

$$\sum_{i \in N \setminus S} x_i + \sum_{i \in S} (1 - x_i) \geq 1 \quad \text{for all } S \subseteq \{1, 2, \dots, n\} \text{ with } |S| \text{ odd} \quad (2a)$$

$$x_i \in [0, 1] \quad \text{for } i = 1, 2, \dots, n \quad (2b)$$

Theorem – Disjunctive program for even parity polytope

P_{even}^n has an extended formulation of size $\mathcal{O}(n^2)$.

Proof:

- ▶ For $k \in \mathbb{Z}$, $P_k := \{x \in [0, 1]^n \mid \sum_{i=1}^n x_i = k\}$ is integral.
- ▶ $P_{\text{even}}^n = \text{conv}(P_0 \cup P_2 \cup \dots \cup P_{2 \cdot \lfloor n/2 \rfloor})$.

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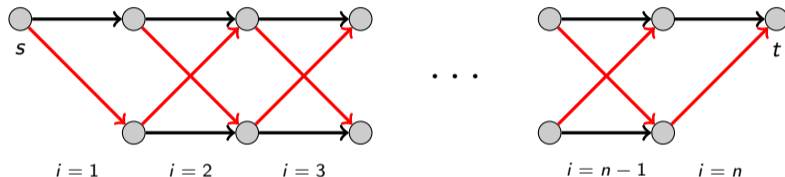
Even Parity Polytope via Flows

Theorem – Flow Extended Formulation

[Carr & Konjevod 2004]

P_{even}^n has an extended formulation of size $4n - 4$.

Construction:



- ▶ Let Q be the s - t -flow polytope of digraph $D = (V, A)$.
- ▶ The vertices of Q are incidence vectors of s - t -paths in D .
- ▶ Define $\pi : \mathbb{R}^A \rightarrow \mathbb{R}^n$ via

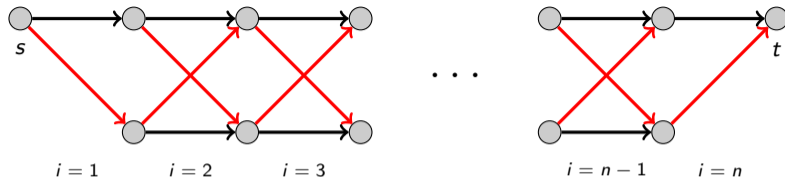
$$\pi(y)_i := \begin{cases} y_{a_i} & \text{for } i = 1 \\ y_{a_i} + y_{b_i} & \text{for } i = 2, 3, \dots, n-1 \\ y_{b_i} & \text{for } i = n. \end{cases}$$

- ▶ The formulation has one inequality per arc.

Claim – Projection

$\pi(Q) = P_{\text{even}}^n$ holds.

Extended Formulation for Even Subsets



Claim – Projection

$\pi(Q) = P_{\text{even}}^n$ holds.

Proof of the claim:

$$\begin{aligned}
 \pi(Q) &= \pi(\text{conv} \{ \chi(W) : W \subseteq A \text{ is } s\text{-}t\text{-path in } D \}) \\
 &= \text{conv} \{ \pi(\chi(W)) : W \subseteq A \text{ is } s\text{-}t\text{-path in } D \} \\
 &= \text{conv} \{ v \in \{0, 1\}^n : \sum_{i=1}^n v_i \in 2\mathbb{Z} \} \\
 &= P_{\text{even}}^n
 \end{aligned}$$

Max-Flow Min-Cut Theorem

Definition – s-t-flow and flow polytope

Let $D = (V, A)$ be a digraph with **source** and **sink** nodes $s, t \in V$, and let $u \in \mathbb{R}_{\geq 0}^A$ be **arc capacities**.

- ▶ An **s-t-flow** is a vector $f \in \mathbb{R}^A$ that satisfies (3).
- ▶ The set of all s-t-flows is called the **s-t-flow polytope** of (D, u) .

Flow constraints:

$$\sum_{a \in \delta^{\text{in}}(v)} f_a - \sum_{a \in \delta^{\text{out}}(v)} f_a = 0 \quad \forall v \in V \setminus \{s, t\}, \quad (3a)$$

$$0 \leq f_a \leq u_a \quad \forall a \in A. \quad (3b)$$

Notation:

$\delta^{\text{out}}(S)$ denotes all arcs $(u, v) \in A$ that have $u \in S$ and $v \notin S$.

$$\delta^{\text{out}}(v) := \delta^{\text{out}}(\{v\}).$$

Definition – s-t-cut

Let $D = (V, A)$ be a digraph with nodes $s, t \in V$. An **s-t-cut** is a cut $\delta^{\text{out}}(S)$ induced by a set $S \subseteq V$ with $s \in S$ and $t \notin S$.

Theorem – Max-Flow Min-Cut Theorem

[Ford & Fulkerson, '62]

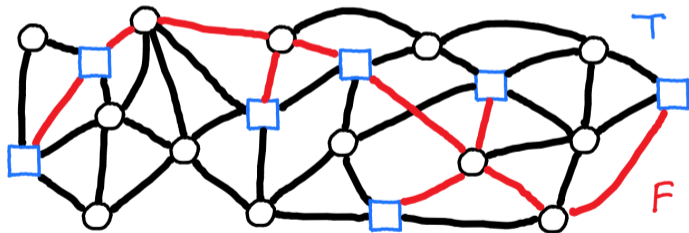
Let $D = (V, A)$ be a digraph with source $s \in V$, sink $t \in V$ and capacities $u \in \mathbb{R}_{\geq 0}^A$. Then the maximum value of an s-t-flow is equal to the minimum capacity of an s-t-cut.

The Steiner Tree Problem

Problem – Steiner tree problem

- ▶ **Input:** Graph $G = (V, E)$, terminals $T \subseteq V$, edge costs $c \in \mathbb{R}_{\geq 0}^E$.
- ▶ **Feasible solutions:** Subsets $F \subseteq E$, called **Steiner trees**, such that (V, F) contains an s - t -path for each pair of terminals $s, t \in T$.
- ▶ **Goal:** Minimize the cost of $c(F) := \sum_{e \in F} c_e$.

Example:



Remark:

- ▶ After removing edges of zero cost, an optimal Steiner tree is indeed a tree, i.e., it is connected and contains no cycles.

Applications:

Prototype problem for network design:

- ▶ telecommunication
- ▶ water/gas supply
- ▶ wiring of a chip
- ▶ ...

The Undirected Cut Formulation for Steiner Trees

Variables:

- ▶ $x_e \in \{0, 1\}$ for $e \in E$: $x_e = 1 \iff e$ belongs to Steiner tree.

IP:

$$\min \sum_{e \in E} c_e x_e \quad (4a)$$

$$\text{s.t.} \quad \sum_{e \in \delta(S)} x_e \geq 1 \quad \forall S \subseteq V : S \cap T \neq \emptyset, T \setminus S \neq \emptyset \quad (4b)$$

$$x \in \{0, 1\}^E \quad (4c)$$

Proposition – Correctness of undirected cut formulation

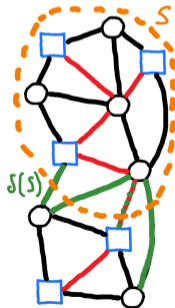
Formulation (4) correctly models the Steiner tree problem.

Proof:

- ▶ Let F be a Steiner tree, $x := \chi(F)$ and let S be as in (4b).
- ▶ Choose $s \in T \cap S$ and $t \in T \setminus S$. There exists an s - t -path P in F .
- ▶ It satisfies $P \cap \delta(S) \neq \emptyset$, showing that (4b) is satisfied.

- ▶ Let x be feasible for (4). We have to show that $F := \text{supp}(x)$ connects T .
- ▶ Suppose there is a pair $s, t \in T$ that is not connected in F .
- ▶ Let $S \subseteq V$ be the set of nodes reachable from s in F . Note: $t \notin S$.
- ▶ By construction, $F \cap \delta(S) = \emptyset$, contradicting (4b). ■

A Steiner cut:



Undirected Extended Flow Formulation for Steiner Trees

Auxiliary data:

- ▶ We fix a **root node** $r \in T$.
- ▶ Let $D = (V, A)$ with $A := \{(u, v) : \{u, v\} \in E\}$ be the **bidirected graph** of G .

Variables:

- ▶ $x_e \in \{0, 1\}$ for $e \in E$: $x_e = 1 \iff e$ belongs to Steiner tree.
- ▶ $f_a^k \in \{0, 1\}$ for $a \in A$ and $k \in T \setminus \{r\}$: f^k models r - k -flow of value 1.

IP:

$$\min \sum_{e \in E} c_e x_e \quad (5a)$$

$$\text{s.t.} \quad \sum_{a \in \delta^{\text{in}}(v)} f_a^k - \sum_{a \in \delta^{\text{out}}(v)} f_a^k = \begin{cases} -1 & \text{if } v = r \\ +1 & \text{if } v = k \\ 0 & \text{otherwise} \end{cases} \quad \forall v \in V, \forall k \in T \setminus \{r\} \quad (5b)$$

$$f_{(u,v)}^k \leq x_{\{u,v\}} \quad \forall (u, v) \in A, \forall k \in T \setminus \{r\} \quad (5c)$$

$$f^k \in \{0, 1\}^A \quad \forall k \in T \setminus \{r\} \quad (5d)$$

$$x \in \{0, 1\}^E \quad (5e)$$

Theorem – Undirected extended flow formulation for Steiner trees

The LP relaxation of (5) is an extended formulation of the LP relaxation of (4) (via projection on the x -variables).

Proof strategy:

- ▶ Define $Q := \{(x, f) \in \mathbb{R}_{\geq 0}^E \times \mathbb{R}^* : (x, f) \text{ satisfies (5b) and (5c)}\}$
and $P := \{x \in \mathbb{R}_{\geq 0}^E : x \text{ satisfies (4b)}\}$.
- ▶ We show $P = \text{proj}_x(Q)$ by first showing $\text{proj}_x(Q) \subseteq P$ and then showing $P \subseteq \text{proj}_x(Q)$.
- ▶ For the first part, we consider some x which is a projection of some vector (x, f) satisfying (5b) and (5c). We will show that x satisfies (4b).
- ▶ For the second part, we consider some $x \in P$, satisfying (4b). We will show that there exists a vector f such that (x, f) satisfy (5b) and (5c).

IP (4):

$$(4b) \quad \sum_{e \in \delta(S)} x_e \geq 1$$

for each $S \subseteq V$
with $S \cap T \neq \emptyset$
and $T \setminus S \neq \emptyset$

IP (5):

$$(5b) \quad r\text{-}k\text{-flows } f^k \in \mathbb{R}^A$$

of value 1 for each
 $k \in T \setminus \{r\}$

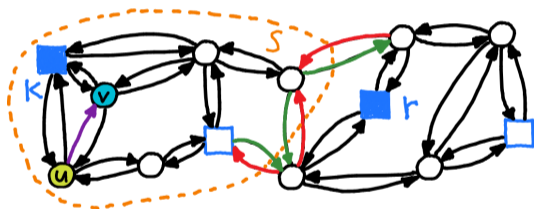
$$(5c) \quad f_{(u,v)}^k \leq x_{\{u,v\}}$$

Cut Relaxation is Contained in Projection of Flow Relaxation

Proof part 1:

- ▶ Let (x, f) satisfy (5b) and (5c).
- ▶ We show that x satisfies (4b) for each $S \subseteq V$ with $r \notin S$ and $T \cap S \neq \emptyset$.
- ▶ Choose $k \in T \cap S$ and consider the sum of (5b) for this k and all $v \in S$:

$$\sum_{v \in S} \left(\sum_{a \in \delta^{\text{in}}(v)} f_a^k - \sum_{a \in \delta^{\text{out}}(v)} f_a^k \right) = 1 + \sum_{v \in S \setminus \{k\}} 0$$



- ▶ Observing that the flow on arcs inside S cancels out, we obtain

$$1 = \sum_{a \in \delta^{\text{in}}(S)} f_a^k - \sum_{a \in \delta^{\text{out}}(S)} f_a^k \leq \sum_{(u,v) \in \delta^{\text{in}}(S)} x_{\{u,v\}} - \sum_{a \in \delta^{\text{out}}(S)} 0 = \sum_{e \in \delta(S)} x_e,$$

which is (4b).

IP (4):

$$(4b) \quad \sum_{e \in \delta(S)} x_e \geq 1$$

for each $S \subseteq V$
with $S \cap T \neq \emptyset$
and $T \setminus S \neq \emptyset$

IP (5):

$$(5b) \quad r\text{-}k\text{-flows } f^k \in \mathbb{R}^A$$

of value 1 for each
 $k \in T \setminus \{r\}$

■

$$(5c) \quad f_{(u,v)}^k \leq x_{\{u,v\}}$$

Projection of Flow Relaxation is Contained in Cut Relaxation

Proof part 2:

- ▶ Let x satisfy (4b).
- ▶ We have to show for each $k \in T \setminus \{r\}$ that there exists an r - k -flow $f^k \in \mathbb{R}_{\geq 0}^A$ with flow value 1 (i.e., f^k satisfies (5b)) that respects arc capacities $x_{\{u,v\}}$ on all arcs $(u, v) \in A$ (i.e., f^k satisfies (5c)).
- ▶ The inequality $\sum_{e \in \delta(S)} x_e \geq 1$ is satisfied for all $S \subseteq V$ with $r \in S$ and $k \notin S$.
- ▶ Hence, the capacity of the minimum r - k -cut is $\gamma \geq 1$.
- ▶ By the Max-Flow Min-Cut Theorem, there exists an r - k -flow of value γ .
- ▶ Scaling this flow by $\frac{1}{\gamma}$ yields a flow of value 1 that also respects the capacities (since we scale down). ■

IP (4):

$$(4b) \quad \sum_{e \in \delta(S)} x_e \geq 1$$

for each $S \subseteq V$
with $S \cap T \neq \emptyset$
and $T \setminus S \neq \emptyset$

IP (5):

$$(5b) \quad r\text{-}k\text{-flows } f^k \in \mathbb{R}^A$$

of value 1 for each
 $k \in T \setminus \{r\}$

$$(5c) \quad f_{(u,v)}^k \leq x_{\{u,v\}}$$

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