UNIVERSITY OF TWENTE.

General-purpose Cutting Planes (Book Chapter 5)

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Topics:

- Chvátal-Gomory Cutting Planes
- Cutting Planes for the Simplex Tableau
- Split Cutting Planes and Lift-and-Project

Preknowledge:

- Polyhedra
- Union of Polyhedra
- Simplex Method

Agenda

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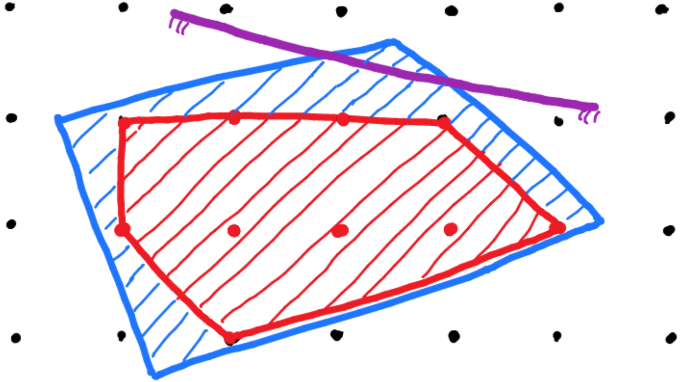
Geometric Idea of Cutting Planes



Definition – Cutting plane

Let $P \subseteq \mathbb{R}^n$ be the LP relaxation of a MIP with integer variables indexed by $I \subseteq [n]$. A **cutting plane** (cut) is an inequality $a^\intercal x \leq \beta$ that

(i) is valid for all $x \in P$ with $x_i \in \mathbb{Z}$ for $i \in I$ (equivalently: valid for P's mixed-integer hull),

(ii) but is **not** valid for P.

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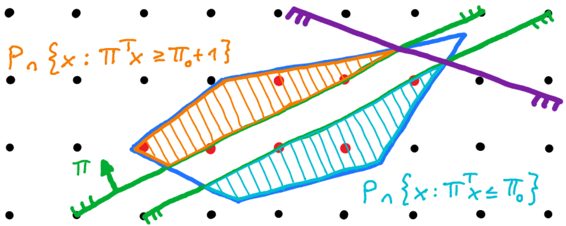
Split Disjunctions

Definition – Split disjunctions and split inequalities

A **split disjunction** is a set $D^{(\pi,\pi_0)} := \{x \in \mathbb{R}^n : \pi^\intercal x \leq \pi_0 \ \vee \ \pi^\intercal x \geq \pi_0 + 1\}$ for some $\pi \in \mathbb{Z}^n \setminus \{\mathbb{O}\}$ and $\pi_0 \in \mathbb{Z}$. For a polyhedron $P \subseteq \mathbb{R}^n$, a valid inequality for $P \cap D^{(\pi,\pi_0)}$ is called a **split inequality** (with respect to $D^{(\pi,\pi_0)}$).

Geometry:



Proposition/Definition – Split cut

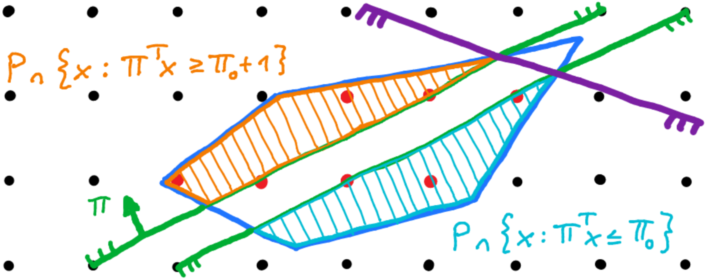
A split inequality that is not valid for P is a cutting plane, called **split cut**.

Split Cuts via Dual Multipliers

Theorem – Split inequalities via dual multipliers [Balas, '74]

Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ with $A \in \mathbb{R}^{m \times n}$. Then an inequality $a^\intercal x \leq \beta$ is a split inequality with respect to the split disjunction $D^{(\pi,\pi_0)}$ if and only if there exist multipliers $u, v \in \mathbb{R}^m$ and $u_0, v_0 \in \mathbb{R}$ satisfying (1).

Geometry:



Inequality system for dual multipliers:

$$a^\intercal = u^\intercal A + u_0 \pi^\intercal \tag{1a}$$
$$\beta \geq u^\intercal b + u_0 \pi_0 \tag{1b}$$
$$u \geq \mathbb{O}_m, \; u_0 \geq 0 \tag{1c}$$

$$a^\intercal = v^\intercal A - v_0 \pi^\intercal \tag{1d}$$
$$\beta \geq v^\intercal b - v_0(\pi_0 + 1) \tag{1e}$$
$$v \geq \mathbb{O}_m, \; v_0 \geq 0 \tag{1f}$$

Split Cuts via Dual Multipliers

Theorem – Split inequalities via dual multipliers **[Balas, '74]**

Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ with $A \in \mathbb{R}^{m \times n}$. Then an inequality $a^\intercal x \leq \beta$ is a split inequality with respect to the split disjunction $D^{(\pi,\pi_0)}$ if and only if there exist multipliers $u, v \in \mathbb{R}^m$ and $u_0, v_0 \in \mathbb{R}$ satisfying (1).

Proof:

- By definition, $a^\intercal x \leq \beta$ is a split inequality if and only if it is valid for $P \cap D^{(\pi,\pi_0)}$.
- This is equivalent to being valid for $P_1 := \{x \in P : \pi^\intercal x \leq \pi_0\}$ and valid for $P_2 := \{x \in P : \pi^\intercal x \geq \pi_0 + 1\}$.
- By strong LP duality, being valid for P_1 is equivalent to

$$\begin{aligned} & \max\{a^\intercal x : x \in P_1\} \leq \beta \\ \iff & \max\{a^\intercal x : Ax \leq b,\ \pi^\intercal x \leq \pi_0\} \leq \beta \\ \iff & \min\{u^\intercal b + u_0\pi_0 : u^\intercal A + u_0\pi^\intercal = a^\intercal,\ u \in \mathbb{R}^m_{\geq 0},\ u_0 \in \mathbb{R}_{\geq 0}\} \leq \beta, \end{aligned}$$

  that is, equivalent to (1a)–(1c).
- Similarly, validity for P_2 is equivalent to (1d)–(1f). ■

Inequality system for dual multipliers:

$$a^\intercal = u^\intercal A + u_0\pi^\intercal \tag{1a}$$
$$\beta \geq u^\intercal b + u_0\pi_0 \tag{1b}$$
$$u \geq \mathbb{O}_m,\ u_0 \geq 0 \tag{1c}$$

$$a^\intercal = v^\intercal A - v_0\pi^\intercal \tag{1d}$$
$$\beta \geq v^\intercal b - v_0(\pi_0 + 1) \tag{1e}$$
$$v \geq \mathbb{O}_m,\ v_0 \geq 0 \tag{1f}$$

Corollary – Disjunctive cut generation LP **[Balas & Ceria, '93]**

The separation problem for split cuts for a given disjunction $D^{(\pi,\pi_0)}$ and a given point $\hat{x} \in \mathbb{R}^n$ reduces to solving the **cut generation LP** (CGLP)

$$\max \hat{x}^\intercal a - \beta \text{ subject to (1).}$$

Proof:

- By the previous theorem, $a^\intercal x \leq \beta$ is a split inequality if and only if there exist u, u_0, v and v_0 feasible for (1).
- Such a solution has positive objective value if and only if the inequality is violated by $\hat{x}$. ∎

Remarks:

- If a split cut exists, the LP is unbounded. In practice, one adds additional **normalization constraints** to make it bounded. Examples:

$$u_0 + v_0 = 1, \qquad \sum_{i=1}^m u_i + \sum_{i=1}^m v_i = 1, \qquad -1 \leq a_j \leq 1 \ \forall j \in [n]$$

Inequality system for dual multipliers:

$$
\begin{aligned}
a^\intercal &= u^\intercal A + u_0 \pi^\intercal && \text{(1a)}\\
\beta &\geq u^\intercal b + u_0 \pi_0 && \text{(1b)}\\
u &\geq \mathbb{O}_m,\ u_0 \geq 0 && \text{(1c)}\\
a^\intercal &= v^\intercal A - v_0 \pi^\intercal && \text{(1d)}\\
\beta &\geq v^\intercal b - v_0(\pi_0 + 1) && \text{(1e)}\\
v &\geq \mathbb{O}_m,\ v_0 \geq 0 && \text{(1f)}
\end{aligned}
$$

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Geometric Idea of Chvátal-Gomory Cuts

Definition / Proposition – Chvátal-Gomory cut **[Chvátal, '73]**

Let $a^\intercal x \leq \beta$ be an inequality valid for a polyhedron $P \subseteq \mathbb{R}^n$, where $a \in \mathbb{Z}^n$. Then

$$a^\intercal x \leq \lfloor \beta \rfloor \tag{2}$$

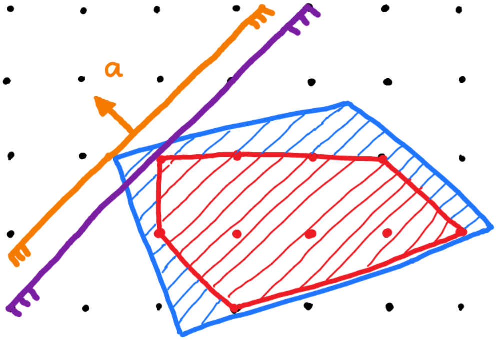
is valid for all $x \in P \cap \mathbb{Z}^n$ and called a **Chvátal-Gomory cut** (CG cut).

Proof:

- For all $x \in \mathbb{Z}^n$, $a^\intercal x \in \mathbb{Z}$.
- Hence, we can round down the right-hand side of the inequality. ■

$a^\intercal x \leq \beta$

$a^\intercal x \leq \lfloor \beta \rfloor$



Notation:

- By $\lfloor \cdot \rfloor$ we denote rounding down to the next integer (for vectors: component-wise).

Separation of Chvátal-Gomory Cuts

Theorem – Complexity of CG cut separation **[Eisenbrand, '99]**

The separation problem for Chvátal-Gomory cuts is coNP-hard.

Theorem – Chvátal-Gomory cut separation for basic solutions **[Gomory, '58]**

Consider a system in standard form $Ax = b$, $x \geq \mathbb{O}$ with $A \in \mathbb{Z}^{m \times n}$. If a basic solution $\hat{x}$ is not integral, then it is violated by a Chvátal-Gomory cut that can be computed in polynomial time.

Proof:

- Let $B \subseteq [n]$ be the basis. We multiply with the inverse of the basis matrix: $x_B + A_{\star,B}^{-1} A_{\star,N} x_N = A_{\star,B}^{-1} b$.
- Since $\hat{x}_N = \mathbb{O}_N$, the fractional variable $\hat{x}_k \notin \mathbb{Z}$ must be basic. Its row reads $x_k + \sum_{j \in N} d_j x_j = \gamma$ for suitable $d \in \mathbb{R}^n$ and $\gamma = \hat{x}_k$.
- By adding $-(d_j - \lfloor d_j \rfloor) x_j \leq 0$ for all $j \in N$, we obtain $x_k + \sum_{j \in N} \lfloor d_j \rfloor x_j \leq \gamma$.
- Since all coefficients are integral, we can derive the following CG cut: $x_k + \sum_{j \in N} \lfloor d_j \rfloor x_j \leq \lfloor \gamma \rfloor$.
- Due to $\hat{x}_N = \mathbb{O}_N$, we have $\hat{x}_k = \gamma > \lfloor \gamma \rfloor$, i.e., $\hat{x}$ violates the cut. ∎

Characterization of Chvátal-Gomory Cuts

Lemma 5.13 – Chvátal-Gomory cuts via dual multipliers

Consider a polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ with $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$. Every CG cut $a^\intercal x \leq \lfloor \beta \rfloor$ is equal to $(\lambda^\intercal A)x \leq \lfloor \lambda^\intercal b \rfloor$ for $\lambda \in \mathbb{R}^m$ with $0 \leq \lambda_i < 1$ for all $i \in [m]$ or implied by such a cut and the original inequalities $Ax \leq b$.

Proof:

- Let $a^\intercal x \leq \lfloor \beta \rfloor$ be a Chvátal-Gomory cut. Hence, $a^\intercal x \leq \beta$ is valid for P.

$$\max\{a^\intercal x : Ax \leq b\} \leq \beta \iff \min\{\mu^\intercal b : \mu^\intercal A = a,\ \mu \in \mathbb{R}^m_{\geq 0}\} \leq \beta \iff \exists \mu \in \mathbb{R}^m_{\geq 0} : \mu^\intercal A = a^\intercal,\ \mu^\intercal b \leq \beta$$

- Define $\lambda \in \mathbb{R}^m_{\geq 0}$ via $\lambda_i := \mu_i - \lfloor \mu_i \rfloor \in [0,1)$ for each $i \in [m]$.
- Since $\lambda^\intercal A = \mu^\intercal A - \lfloor \mu \rfloor^\intercal A = a^\intercal - \lfloor \mu \rfloor^\intercal A$ is an integer vector, λ defines the CG cut $\lambda^\intercal A x \leq \lfloor \lambda^\intercal b \rfloor$.
- The sum of this inequality plus $\lfloor \mu_i \rfloor$ times $A_{i,\star} x \leq b_i$ for all $i \in [m]$ reads

$$\lambda^\intercal A x + \lfloor \mu \rfloor^\intercal A x \leq \lfloor \lambda^\intercal b \rfloor + \lfloor \mu \rfloor^\intercal b.$$

- The left-hand side is $\lambda^\intercal A x + \lfloor \mu \rfloor^\intercal A x = (\mu - \lfloor \mu \rfloor + \lfloor \mu \rfloor)^\intercal A x = \mu^\intercal A x = a^\intercal x$.
- The right-hand side is $\lfloor \lambda^\intercal b \rfloor + \lfloor \mu \rfloor^\intercal b = \lfloor (\mu - \lfloor \mu \rfloor)^\intercal b \rfloor + \lfloor \mu \rfloor^\intercal b = \lfloor \mu^\intercal b - \lfloor \mu \rfloor^\intercal b \rfloor + \lfloor \mu \rfloor^\intercal b = \lfloor \mu^\intercal b \rfloor \leq \lfloor \beta \rfloor$.
- Hence, the given CG cut is implied by $\lambda^\intercal A x \leq \lfloor \lambda^\intercal b \rfloor$ together with $Ax \leq b$. $\blacksquare$

Finiteness of Chvátal-Gomory Closure

Lemma 5.13 – Chvátal-Gomory cuts via dual multipliers

Consider a polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ with $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$. Every CG cut $a^\intercal x \leq \lfloor \beta \rfloor$ is equal to $(\lambda^\intercal A)x \leq \lfloor \lambda^\intercal b \rfloor$ for $\lambda \in \mathbb{R}^m$ with $0 \leq \lambda_i < 1$ for all $i \in [m]$ or implied by such a cut and the original inequalities $Ax \leq b$.

Definition – Chvátal-Gomory closure

The **Chvátal-Gomory closure** of a polyedron P is the set of points that satisfy all Chvátal-Gomory cuts.

Theorem 5.14 – Finiteness of Chvátal-Gomory closure **[Chvátal '73]**

The Chvátal-Gomory closure of a rational polyhedron P is again a rational polyhedron.

Proof:

- By Lemma 5.13 we only need to consider Chvátal-Gomory cuts with multipliers in $[0, 1)$.
- Since $\{\lambda^\intercal A \in \mathbb{R}^n : 0 \leq \lambda_i < 1 \ \forall i \in [m]\}$ is a bounded set, the set $\{\lambda^\intercal A \in \mathbb{Z}^n : 0 \leq \lambda_i < 1 \ \forall i \in [m]\}$ is finite.
- Hence, only finitely many inequalities are non-redundant. ■

Separation of Chvátal-Gomory Cuts via MIP

Variables:

- $\lambda_i \in [0,1]$ for each $i \in [m]$: dual multiplier for inequality $A_{i,\star} x \leq b_i$.
- $a_j \in \mathbb{Z}$ for each $j \in [n]$: coefficient of CG cut.
- $\delta \in \mathbb{Z}$: right-hand side of CG cut.

Proposition – Correctness of formulation (3)

Consider the inequality system $Ax \leq b$, a point $\hat{x} \in \mathbb{R}^n$ and a parameter $0 < \varepsilon < 1$. Then formulation (3) yields a CG cut $a^\intercal x \leq \delta$ with maximum violation (with respect to $\hat{x}$) that is at most $1 - \varepsilon$.

MIP:

$$
\begin{aligned}
\max \quad & \sum_{j=1}^{n} \hat{x}_j a_j - \delta && \text{(3a)}\\
\text{s.t.} \quad & A^\intercal \lambda - a = \mathbb{O} && \text{(3b)}\\
& b^\intercal \lambda - \delta \leq 1 - \varepsilon && \text{(3c)}\\
& \lambda \in [0,1]^m && \text{(3d)}\\
& a \in \mathbb{Z}^n && \text{(3e)}\\
& \delta \in \mathbb{Z} && \text{(3f)}
\end{aligned}
$$

Proof:

- A feasible solution (λ, a, δ) yields the valid inequality $a^\intercal x \leq b^\intercal \lambda$ with integral coefficient vector $a \in \mathbb{Z}^n$.
- Due to $\varepsilon > 0$, $\delta \geq b^\intercal \lambda - 1 + \varepsilon$ and $\delta \in \mathbb{Z}$ we have $\delta \geq \lfloor b^\intercal \lambda \rfloor$.
- The objective (3a) is the violation of $a^\intercal x \leq \delta$ with respect to $\hat{x}$. ■

Lesson Recap – **Any Questions?**

1. Geometry of Cutting Planes
   - Geometric Idea
2. Split Cutting Planes
   - Split Disjunctions and Split Cuts
   - Lift-and-Project Cut Generation
3. Chvátal-Gomory Cutting Planes
   - Geometric Idea
   - Separation from the Simplex Tableau
   - Cut Closure