UNIVERSITY OF TWENTE.

Problem-specific Cutting Planes (Book Section 7.4)

Matthias Walter

Topics:

- Subtour Formulation of the TSP
- Dimension / Facets of the TSP Polytope
- Comb Inequalities

Preknowledge:

- Polyhedra
- Hamiltonian cycles & cuts in graphs

Agenda

1. The Subtour Formulation for the TSP
   - Subtour Formulation
   - Separation Algorithm
2. The TSP Polytope
   - Dimension
   - Subtour Facets
3. Cutting Planes
   - 2-Matching Inequalities
   - Comb Inequalities
   - Further Inequalities

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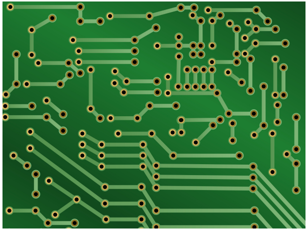
Traveling Salesperson Problem

Problem – Traveling Salesperson Problem (TSP)

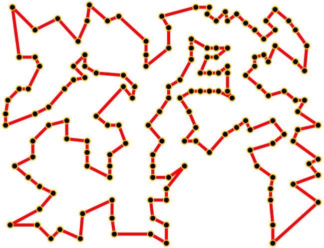
- **Input:** Graph $G = (V, E)$, edge costs $c \in \mathbb{R}^E_{\geq 0}$.
- **Feasible solutions:** Hamiltonian cycles $T \subseteq E$, called **tours**.
- **Objective:** Minimize tour cost $c(T) := \sum_{e \in T} c_e$.

Goal: drill all holes in shortest time



Mathematical modelling

TSP: find shortest tour through all points in network



Theorem – Hardness of TSP [Karp, '72]

The Traveling Salesperson Problem is NP-hard.

Subtour Formulation

Problem – Traveling Salesperson Problem (TSP)

- **Input:** Graph $G = (V, E)$, edge costs $c \in \mathbb{R}^E_{\geq 0}$.
- **Feasible solutions:** Hamiltonian cycles $T \subseteq E$, called **tours**.
- **Objective:** Minimize tour cost $c(T) := \sum_{e \in T} c_e$.

Variables:

- $x \in \{0,1\}^E$: $x_e = 1$ if and only if e is part of the tour.

IP:

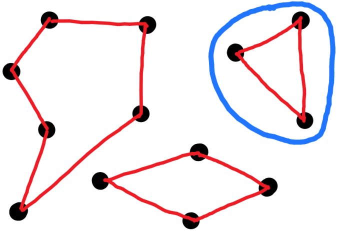
$$\min \sum_{e \in E} c_e x_e \tag{1a}$$

$$\text{s.t.} \sum_{e \in \delta(v)} x_e = 2 \qquad \forall v \in V \tag{1b}$$

$$\sum_{e \in E[S]} x_e \leq |S| - 1 \qquad \forall S \subset V,\ 2 \leq |S| \leq |V| - 2 \tag{1c}$$

$$x_e \in \{0,1\} \qquad \forall e \in E \tag{1d}$$

Subtours:



Separation Problem

Problem – Separation problem for subtour inequalities

- **Input:** Graph $G = (V, E)$, vector $\hat{x} \in [0,1]^E$ satisfying the degree constraints.
- **Goal:** Find a violated inequality: $\sum_{e \in E[S]} \hat{x}_e > |S| - 1$ (for $2 \leq |S| \leq |V| - 2$)

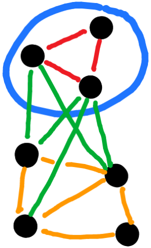
  or assert that $\hat{x}$ satisfies all subtour constraints.

Solution:

- The separation problem can be reduced to a minimum cut problem:
- We subtract $\sum_{e \in E[S]} \hat{x} > |S| - 1$ from $\sum_{v \in S} \frac{1}{2} \sum_{e \in \delta(v)} \hat{x}_e = 2 \cdot \frac{1}{2} \cdot |S|$, which yields

$$\frac{1}{2} \sum_{e \in \delta(S)} \hat{x}_e < 1 \iff \sum_{e \in \delta(S)} \hat{x}_e < 2.$$

- Hence, it suffices to find a minimum (nontrivial) cut $\delta(S)$ with respect to edge capacities $\hat{x}_e$.
- This can be done in time $\mathcal{O}(|V| \cdot |E| + |V|^2 + \log |V|)$ time using the algorithm of Nagomoshi & Ibaraki ('92) and Stoer & Wagner ('97). ■



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The TSP Polytope and its Dimension

Definition – TSP polytope

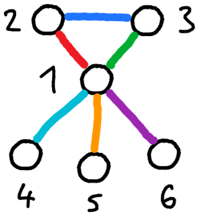
The **TSP polytope** of $G = (V, E)$ is defined as $P_{\text{tsp}}(G) := \text{conv}\{\chi(T) : T \text{ tour in } G\}$.

Theorem 7.18 – Dimension of the TSP polytope of a complete graphs [Grötschel & Padberg '79]

Let $K_n = (V_n, E_n)$ be the complete graph with $n \geq 3$ nodes. Then $\dim(P_{\text{tsp}}(K_n)) = |E_n| - |V_n| = \binom{n}{2} - n$.

Basis of system (1b):

- We assume $V_n = \{1, 2, \dots, n\}$.
- Let $B := \delta(1) \cup \{\{2,3\}\}$ be the star cut of node 1 plus an edge.
- The submatrix indexed by nodes 1, 2 and 3, and edges $\{1,2\}$, $\{1,3\}$ and $\{2,3\}$ has full rank.
- Adding node $i \in \{4, 5, \dots, n\}$ and connecting it with a single edge maintains full rank because the row of the node yields a unit vector.



$$\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Proof of upper bound:

- Since the variables of B induce an $n \times n$ submatrix of full rank, the equations are linearly independent.

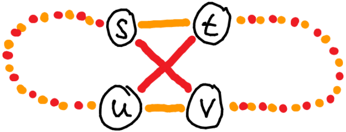
Dimension Proof (continued)

Theorem 7.18 – Dimension of the TSP polytope of a complete graphs [Grötschel & Padberg '79]

Let $K_n = (V_n, E_n)$ be the complete graph with $n \geq 3$ nodes. Then $\dim(P_{\text{tsp}}(K_n)) = |E_n| - |V_n| = \binom{n}{2} - n$.

Proof of lower bound using generic equation approach:

- Let $\sum_{e \in E} c_e x_e = \gamma$ be an equation that is valid for $P_{\text{tsp}}(K_n)$.
- Since the edge set $B = \delta(1) \cup \{\{2,3\}\}$ is a basis, we can combine equations (1b) linearly such that the coefficients of the combined equation agree with c_e for all $e \in B$.
- We subtract it from $c^\intercal x = \gamma$, which yields $c'^\intercal x = \gamma'$ with $c'_e = 0$ for all $e \in B$.
- Consider four arbitrary nodes $s, t, u, v \in [n]$. Let T_1 be a tour that traverses these nodes in the order s, t, v, u (other nodes inbetween allowed). Then $T_2 := T_1 \setminus \{\{s,t\},\{u,v\}\} \cup \{\{s,v\},\{t,u\}\}$ is also a tour.
- From $c'^\intercal \chi(T_i) = \gamma'$ for $i = 1, 2$ we obtain $0 = c'^\intercal(\chi(T_1) - \chi(T_2)) = c'_{\{s,t\}} + c'_{\{u,v\}} - c'_{\{s,v\}} - c'_{\{t,u\}}$.



- For $i \in \{4, 5, \dots, n\}$, this implies $0 = c'_{\{2,i\}} + c'_{\{1,3\}} - c'_{\{1,i\}} - c'_{\{2,3\}} = c'_{\{2,i\}}$.
- For $i, j \in \{3, 4, \dots, n\}$ with $i \neq j$: $0 = c'_{\{1,2\}} + c'_{\{i,j\}} - c'_{\{2,i\}} - c'_{\{1,j\}} = c'_{\{i,j\}}$.
- Hence, $c' = \mathbb{O}$ and thus c is a linear combination of equations from (1b). ■

Subtour Inequalities Define Facets (1)

Theorem 7.19 – Subtour inequalities define facets of the TSP polytope [Grötschel & Padberg '79]

Let $K_n = (V_n, E_n)$ be the complete graph with $n \geq 4$ nodes and let $S \subseteq V$ with $3 \leq |S| \leq n - 3$. Then inequalities (1c) are facet-defining for $P_{\text{tsp}}(K_n)$.

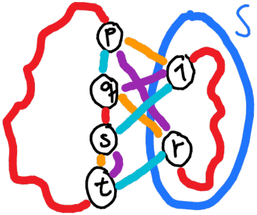
Proof using generic equation approach:

- Let F be a face defined by $\sum_{e \in E[S]} x_e \leq |S| - 1$ and let $\bar{F}$ be a facet, defined by $\sum_{e \in E} c_e x_e \leq \gamma$, with $F \subseteq \bar{F}$.
- W.l.o.g. we assume $1 \in S$ and $2, 3 \in V \setminus S$. Since the edge set $B = \delta(1) \cup \{\{2,3\}\}$ is a basis of the equation system (1b), we can combine these linearly such that the coefficients of the combined equation agree with c_e for all $e \in B \setminus E[S]$ and with $(c_e - 1)$ for all $e \in B \cap E[S]$.
- We subtract it from $c^\intercal x \leq \gamma$, which yields the equivalent inequality $c'^\intercal x \leq \gamma'$ with $c'_e = 0$ for all $e \in B \setminus E[S]$ and $c'_e = 1$ for all $e \in B \cap E[S]$.
- For distinct edges $\{p,q\}, \{s,t\} \in E[V \setminus S]$ and $r \in S \setminus \{1\}$ we consider a Hamiltonian 1-r-path P in $E[S]$, a Hamiltonian cycle C in $E[V \setminus S]$ traversing $\{p,q\}, \{s,t\}$ and visiting p, q, s and t in that order. Define

$$T_1 := P \cup C \setminus \{\{p,q\}\} \cup \{\{1,p\},\{r,q\}\},$$

$$T_2 := P \cup C \setminus \{\{p,q\}\} \cup \{\{1,q\},\{r,p\}\}, \text{ and}$$

$$T_3 := P \cup C \setminus \{\{s,t\}\} \cup \{\{1,s\},\{r,t\}\}$$



Subtour Inequalities Define Facets (2)

Theorem 7.19 – Subtour inequalities define facets of the TSP polytope [Grötschel & Padberg '79]

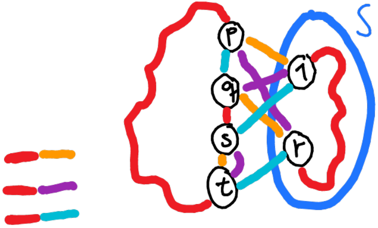
Let $K_n = (V_n, E_n)$ be the complete graph with $n \geq 4$ nodes and let $S \subseteq V$ with $3 \leq |S| \leq n-3$. Then inequalities (1c) are facet-defining for $P_{\text{tsp}}(K_n)$.

Tours: $T_1 := P \cup C \setminus \{\{p,q\}\} \cup \{\{1,p\},\{r,q\}\}$, $T_2 := P \cup C \setminus \{\{p,q\}\} \cup \{\{1,q\},\{r,p\}\}$

$T_3 := P \cup C \setminus \{\{s,t\}\} \cup \{\{1,s\},\{r,t\}\}$.

Proof (continued):

- Observe $\chi(T_i) \in F \subseteq \bar{F}$ for $i = 1,2,3$.
- From $c'(T_1) = \gamma' = c'(T_2)$ we obtain $-c'_{p,q} + c'_{1,p} + c'_{r,q} = -c'_{p,q} + c'_{1,q} + c'_{r,p}$.
- From $c'_{1,p} = 0$ and $c'_{1,q} = 0$ we obtain $c'_{r,q} = c'_{r,p}$ (for each $r \in S$ and all $p,q \in V \setminus S$).
- Similarly, $c'(T_1) = \gamma' = c'(T_3)$ yields $-c'_{p,q} + c'_{1,p} + c'_{r,q} = -c'_{s,t} + c'_{1,s} + c'_{r,t}$.
- From $c'_{1,s} = c'_{1,p}$ and $c'_{r,q} = c_{r,t}$ we obtain $c'_{p,q} = c'_{s,t}$ (for every edge pair $\{p,q\},\{s,t\}$).
- From $c'_{2,3} = 0$ we conclude that $c'_e = 0$ for all $e \in E[V \setminus S]$ and, for each $s \in S$, $c'_{s,t}$ is the same for all $t \in V \setminus S$.



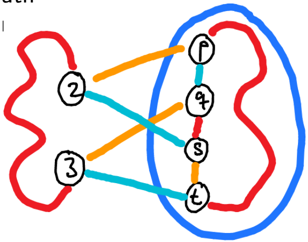
Subtour Inequalities Define Facets (3)

Theorem 7.19 – Subtour inequalities define facets of the TSP polytope [Grötschel & Padberg '79]

Let $K_n = (V_n, E_n)$ be the complete graph with $n \geq 4$ nodes and let $S \subseteq V$ with $3 \leq |S| \leq n-3$. Then inequalities (1c) are facet-defining for $P_{\text{tsp}}(K_n)$.

Proof (continued):

- We already proved that $c'_e = 0$ for all $e \in E[V \setminus S]$ and, for each $s \in S$, $c'_{s,t}$ is the same for all $t \in V \setminus S$.
- For distinct edges $\{p, q\}, \{s, t\} \in E[S]$ we consider a Hamiltonian 2-3-path P in $E[V \setminus S]$, a Hamiltonian cycle C in $E[S]$ traversing $\{p, q\}, \{s, t\}$ a visiting p, q, s and t in that order.
- Define $T_1 := P \cup C \setminus \{\{p, q\}\} \cup \{\{2, p\}, \{3, q\}\}$ and $T_2 := P \cup C \setminus \{\{s, t\}\} \cup \{\{2, s\}, \{3, t\}\}$.
- Again, $\chi(T_i) \in F \subseteq \bar{F}$ holds for $i = 1, 2$.
- From $c'(T_1) = \gamma' = c'(T_2)$ we obtain $-c'_{p,q} + c'_{2,p} + c'_{3,q} = -c'_{s,t} + c'_{2,s} + c'_{3,t}$.
- For $q = s = 1$ we have $c'_{p,q} = c'_{s,t} = 0$ and thus $c'_{2,p} = c'_{3,t}$, which proves that c'_e is the same for all $e \in \delta(S)$ and equal to 0 due to $c'_{1,2} = 0$.
- Thus, $-c'_{p,q} = -c'_{s,t}$ for all $p, q, s, t \in S$, i.e, $c'^{\intercal} x = \sum_{e \in E[S]} x_e$ after scaling.
- It is easily checked that also $\gamma' = |S| - 1$ holds. ■



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Perfect Simple 2-Matchings

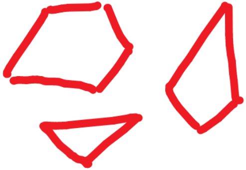
Definition – (Perfect) simple 2-matching

Let $G = (V, E)$. A subset $M \subseteq E$ of edges is called a **simple 2-matching** if every node has degree at most 2 in M. A simple 2-matching M is called **perfect** if $|M| = |V|$.

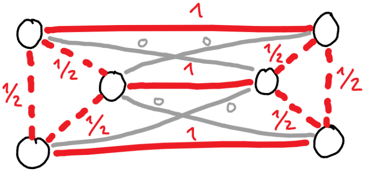
Remark:

- Perfect simple 2-matchings are precisely those edge subsets that consist of disjoint cycles that cover every node.
- The attribute **simple** refers to the requirement that every edge can be used at most once.
- An IP formulation is given by the degree constraints (1b) and the variable domains (1d).

A perfect simple 2-matching:



A vertex of the LP relaxation:



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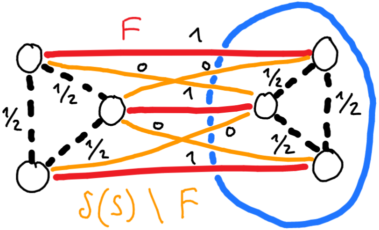


Blossom inequalities:

$$\sum_{e \in \delta(S) \setminus F} x_e + \sum_{e \in F} (1 - x_e) \geq 1 \quad \forall F \subseteq \delta(S),\ |F| \text{ odd},\ S \subseteq V \tag{2}$$

Theorem – Perfect formulation for perfect simple 2-matchings [Edmonds '65]

The formulation consisting of degree constraints (1b), variable bounds (1d) and (2) is a perfect formulation for perfect simple 2-matchings.

Theorem – Separation problem for (2) [Padberg & Rao '82]

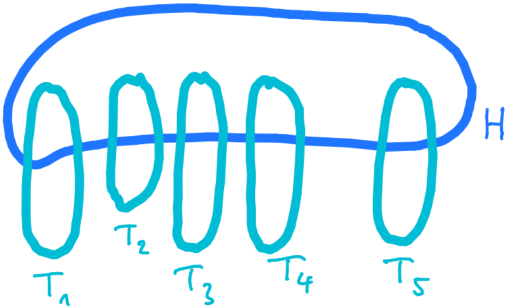
The separation problem for (2) can be solved in polynomial time.

Comb Inequalities

Definition – Comb inequalities [Grötschel & Padberg '79]

A **comb inequality** (3) is defined by a **handle** $H \subseteq V$ and and odd number $t \geq 3$ of **teeth** $T_1, T_2, \dots, T_t \subseteq V$ satisfying

- Teeth are attached to the handle: $T_i \cap H \neq \varnothing$ for $i = 1, 2, \dots, t$.
- Teeth are not contained in the handle: $T_i \setminus H \neq \varnothing$ for $i = 1, 2, \dots, t$.
- Teeth are disjoint: $T_i \cap T_j = \varnothing$ for all $i \neq j$.



Comb inequality:

$$\sum_{e \in \delta(H)} x_e + \sum_{i=1}^{t} \sum_{e \in \delta(T_i)} x_e \geq 3t + 1 \qquad (3)$$

Validity and Facetness of Comb Inequalities

Comb inequality:

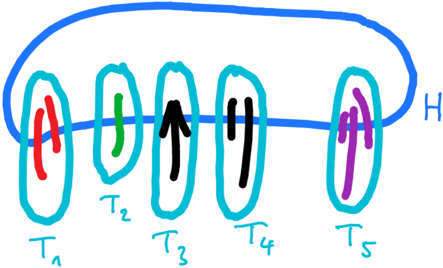
$$\sum_{e \in \delta(H)} x_e + \sum_{i=1}^{t} \sum_{e \in \delta(T_i)} x_e \geq 3t + 1 \qquad (3)$$

Theorem – Validity and facetness of comb inequalities [Grötschel & Padberg '79]

Let $G = (V, E)$ be a graph. Then (3) is valid for $P_{\text{tsp}}(G)$. If G is a complete graph, then it is facet-defining.

Proof of validity:

- Let T be a tour and let x be its incidence vector.
- Define $d_i(H) := \{e \in E : e \cap (T_i \cap H) \neq \varnothing,\ e \cap (T_i \setminus H) \neq \varnothing\}$
- Note that the $d_i(H)$ are disjoint.
- We have $\delta(H) \supseteq d_1(H) \cup d_2(H) \cup \cdots \cup d_t(H)$.
- For $i \in \{1, 2, \dots, t\}$ we have $|T \cap \delta(T_i)| \geq 3$ or $|T \cap \delta(T_i)| = 2$.
- In the latter case, T contains an edge from $T_i \setminus H$ to $T_i \cap H$.
- In any case we have $|T \cap d_i(H)| + |T \cap \delta(T_i)| \geq 3$.
- Summing up yields $\sum_{e \in \delta(H)} x_e + \sum_{i=1}^{t} \sum_{e \in \delta(T_i)} x_e \geq 3t$.
- The left-hand side is even, but $3t$ is odd, so we can add 1. ■



Separation of Comb Inequalities

Theorem – Separation of for fixed number of teeth **[Carr '97]**

For a fixed number t, the comb inequalities with t teeth can be separated by solving $\mathcal{O}(n^{2t})$ maximum flow problems.

Theorem – Separation of maximally violated combs **[Fleischer & Tardos '99]**

For planar G, one can find a comb inequality that is violated maximally (i.e., by $1/2$), if such an inequality exists, in time $\mathcal{O}(n^2 \log n)$.

Theorem – Separaton for fixed handle **[Caprara & Letchford '01]**

For a fixed handle H, the separation problem for $\{0, 1/2\}$-cuts (all Chvátal-Gomory cuts with multipliers in $\{0, 1/2\}$, a superclass of comb inequalities) can be solved in polynomial time.

Theorem – Separaton of simple comb inequalities **[Letchford & Lodi '02]**

The separation problem for **simple comb inequalities** (each tooth T has $|T \cap H| = 1$ or $|T \setminus H| = 1$) can be solved in polynomial time.

Remark:

The computational complexity of the comb separation problem is still unknown.

Generalizations of Combs

Theorem – Clique-tree inequalities are facet-defining **[Grötschel & Pulleyblank '86]**

The **clique-tree inequalities** generalize subtour and comb inequalities and are facet-defining for the TSP polytope.

Theorem – Domino-parity inequalities **[Letchford '00]**

Domino-parity inequalities generalize comb inequalities and can be separated in $\mathcal{O}(n^3)$ if G is planar.

Lesson Recap – **Any Questions?**

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