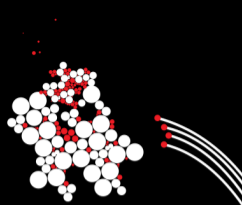


Matthias Walter

Scheduling with Integer Programming and Project

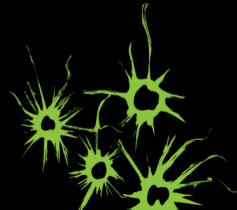


Topics:

- ▶ Original-space models
- ▶ Extended formulations
- ▶ Time discretization

Preknowledge:

- ▶ Polyhedra



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- 1 Single Machine Scheduling
 - Single Machine Scheduling Problem
 - Original Space Description
 - Extended Formulation
 - Time Discretization

- 2 Project (Last Assignment)
 - Description

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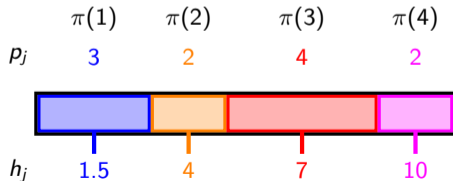
Single Machine Scheduling Problem

Definition – Permutation, half times

Consider a set J of jobs J with processing times $p_j > 0$ for each $j \in J$. A schedule of all jobs on a single machine without idle times is determined by its **permutation** $\pi : \{1, 2, \dots, n\} \rightarrow J$ which assigns a position in the schedule to the job. For a permutation π , the **half time vector** $h^\pi \in \mathbb{R}^J$ specifies when half of each job is carried out. It is the average of the **starting time vector** and the **completion time vector**.

Proposition – Half times for a permutation

For a permutation π , the half times are given by $h_{\pi(k)}^\pi = \sum_{\ell < k} p_{\pi(\ell)} + \frac{1}{2} p_{\pi(k)}$.



Smith's Rule

Theorem – Smith's Rule

[Smith '56]

Let $w \in \mathbb{R}^J$. A schedule minimizes $w(h^\pi) := \sum_{j \in J} w_j h_j^\pi$ if and only if

$$\frac{w_i}{p_i} \geq \frac{w_j}{p_j} \text{ holds for all } i, j \in J \text{ with } h_i^\pi < h_j^\pi. \quad (1)$$

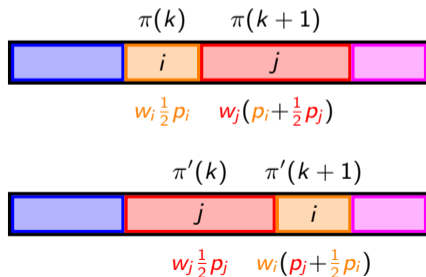
Proof:

- ▶ Suppose (1) is not satisfied for some permutation π . Then there must exist jobs i and j that violate (1) and are scheduled directly after another.
- ▶ Let π' be equal to π , except that $\pi(k) = i$, $\pi(k+1) = j$ but $\pi'(k) = j$ and $\pi'(k+1) = i$ hold for a suitable position k .
- ▶ The objective value change is

$$\begin{aligned} w(h^{\pi'}) - w(h^\pi) &= w_i(h_i^{\pi'} - h_i^\pi) + w_j(h_j^{\pi'} - h_j^\pi) \\ &= w_i \left(p_j + \frac{p_i}{2} - \frac{p_i}{2} \right) + w_j \left(\frac{p_j}{2} - p_i - \frac{p_j}{2} \right) = w_i p_j - w_j p_i < 0. \end{aligned}$$

- ▶ This contradicts optimality of π .
- ▶ Reverse direction: swapping jobs with “=” in (1) does not change the objective value. ■

Example:



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Definition – Half time polytope

The **half time polytope** is the convex hull of all half time vectors

$$P_{\text{ht}} := \text{conv} \left\{ h^\pi \in \mathbb{R}_+^J : \pi \text{ permutation} \right\}.$$

Some constraints:

$$\sum_{j \in S} p_j x_j \geq g(S) \text{ for all } \emptyset \neq S \subsetneq J \quad (2a)$$

$$\sum_{j \in J} p_j x_j = g(J) \quad (2b)$$

► where $g(S) := \min \left\{ \sum_{j \in S} p_j x'_j : x' \in P_{\text{ht}} \right\}$ for $S \subseteq J$.

The Value of $g(S)$

Corollary – Consequence of Smith's Rule

We have $g(S) := \min \left\{ \sum_{j \in S} p_j x'_j : x' \in P_{\text{ht}} \right\} = \frac{1}{2} p(S)^2$ for every $S \subseteq J$.

Proof:

- ▶ We minimize with weights $w_j := p_j$ for $j \in S$ and $w_j := 0$ for $j \notin S$.
- ▶ By Smith' Rule, we have to sort the jobs by w_j/p_j , which is either 1 ($j \in S$) or 0 ($j \notin S$).
- ▶ The objective value of any such permutation is

$$\begin{aligned} & w_{s_1} \frac{1}{2} p_{s_1} + w_{s_2} (p_{s_1} + \frac{1}{2} p_{s_2}) + \dots + w_{s_k} (p_{s_1} + \dots + p_{s_{k-1}} + \frac{1}{2} p_{s_k}) + 0 \\ &= p_{s_1} \frac{1}{2} p_{s_1} + p_{s_2} (p_{s_1} + \frac{1}{2} p_{s_2}) + \dots + p_{s_k} (p_{s_1} + \dots + p_{s_{k-1}} + \frac{1}{2} p_{s_k}) \\ &= \frac{1}{2} (p_{s_1} + p_{s_2} + \dots + p_{s_k})^2 = \frac{1}{2} p(S)^2. \quad \blacksquare \end{aligned}$$

Constraints:

$$(2a): \sum_{j \in S} p_j x_j \geq g(S) \\ \forall \emptyset \neq S \subsetneq J$$

$$(2b): \sum_{j \in J} p_j x_j = g(J)$$

Corollary – Validity of inequalities

Every half time vector h^π satisfies (2).

Proof:

- ▶ The \geq -inequalities are satisfied by definition.
- ▶ The minimum in $g(J)$ is attained by all h^π , so (2b) holds. \blacksquare

Lemma – Supermodularity

$g = \frac{1}{2}p(S)^2$ is supermodular, i.e., $g(S \cap T) + g(S \cup T) \geq g(S) + g(T) \quad \forall S, T \subseteq J$.

Proof:

$$\begin{aligned}g(S \cap T) + g(S \cup T) &= \frac{1}{2}p(S \cap T)^2 + \frac{1}{2}p(S \cup T)^2 = \frac{1}{2}p(S \cap T)^2 + \frac{1}{2}(p(S \setminus T) + p(T \setminus S) + p(S \cap T))^2 \\ &= \frac{2}{2}p(S \cap T)^2 + \frac{1}{2}p(S \setminus T)^2 + \frac{1}{2}p(T \setminus S)^2 + p(S \setminus T) \cdot p(T \setminus S) \\ &\quad + p(S \setminus T) \cdot p(S \cap T) + p(T \setminus S) \cdot p(S \cap T) \\ &= \frac{1}{2}p(S)^2 + \frac{1}{2}p(T)^2 + p(S \setminus T) \cdot p(T \setminus S) \geq g(S) + g(T). \quad \blacksquare\end{aligned}$$

Lemma – Uncrossing

Let $x \in \mathbb{R}^J$ satisfy (2) for $S \cup T$ and for $S \cap T$, and assume it satisfies (2) for S and for T with equality. Then $S \subseteq T$ or $T \subseteq S$.

Proof:

► Coefficient-wise comparison, the assumptions and the proof above imply:

$$0 = \sum_{j \in S \cap T} p_j x_j + \sum_{j \in S \setminus T} p_j x_j - \sum_{j \in S} p_j x_j - \sum_{j \in T} p_j x_j \geq g(S \cap T) + g(S \cup T) - g(S) - g(T) = p(S \setminus T)p(T \setminus S).$$

Theorem – Half time polytope

[Queyranne '91]

The half time polytope is the set of vectors $x \in \mathbb{R}^J$ that satisfy (2).

Proof:

- ▶ Let $x \in \mathbb{R}^J$ be a vertex of (2).
- ▶ Hence, there exist (at least) n tight constraints (2) corresponding to sets $S_1, S_2, \dots, S_n \subseteq J$.
- ▶ By the uncrossing lemma, these sets form a chain, i.e., $S_1 \subsetneq S_2 \subsetneq \dots \subsetneq S_n$.
- ▶ Since $S_1 \neq \emptyset$, we have $S_k = \{j_1, j_2, \dots, j_k\}$ for all $k \in \{1, 2, \dots, n\}$, where $J = \{j_1, j_2, \dots, j_n\}$.
- ▶ Tightness of the inequalities yields $p_{j_1} x_{j_1} = g(\{j_1\}) = \frac{1}{2} p_{j_1}^2$ and thus $x_{j_1} = \frac{1}{2} p_{j_1}$.
- ▶ For S_2 we have $p_{j_2} x_{j_2} = g(\{j_1, j_2\}) - g(\{j_1\})$ follows.
- ▶ By induction, also $x_{j_k} = \sum_{i < k} p_{j_i} \frac{1}{2} p_{j_k}$ follows for $k \in \{2, 3, \dots, n\}$.
- ▶ We conclude that x is a half time vector. ■

Separation Problem

Reminder: $\sum_{j \in S} p_j x_j \geq \min\{\sum_{j \in S} p_j x'_j : x' \in P_{\text{ht}}\}$ for all $S \subseteq J$ (2)

Theorem – Separation of inequalities

[Queyranne '91]

The separation problem for (2) can be solved in $\mathcal{O}(n \log n)$ time.

Proof:

- ▶ A point $\hat{x} \in \mathbb{R}^J$ violates (2a) if and only if there exists a set $S \subseteq J$ with

$$\Gamma(S) := g(S) - \sum_{j \in S} p_j \hat{x}_j > 0.$$

- ▶ Add an element $k \notin S$ to S yields

$$\begin{aligned} \Gamma(S \cup \{k\}) &= \frac{1}{2} p(S \cup \{k\})^2 - \sum_{j \in S \cup \{k\}} p_j \hat{x}_j \\ &= \frac{1}{2} p(S)^2 + p(S)p_k + \frac{1}{2} p_k^2 - \sum_{j \in S} p_j \hat{x}_j - p_k \hat{x}_k \\ &= \Gamma(S) + p_k(p(S) + \frac{1}{2} p_k - \hat{x}_k). \end{aligned}$$

- ▶ Thus, if S is a maximizer and $k \notin S$, this implies $p(S) + \frac{1}{2} p_k - \hat{x}_k \leq 0$, which is equivalent to $\hat{x}_k \geq p(S) + \frac{1}{2} p_k$.

Theorem – Separation of inequalities

[Queyranne '91]

The separation problem for (2) can be solved in $\mathcal{O}(n \log n)$ time.

Proof (continued):

- ▶ Similarly, removing an element $k \in S$ yields

$$\begin{aligned} \Gamma(S \setminus \{k\}) &= \frac{1}{2}p(S \setminus \{k\})^2 - \sum_{j \in S \setminus \{k\}} p_j \hat{x}_j \\ &= \frac{1}{2}p(S)^2 - \frac{1}{2}p_k^2 - p(S \setminus \{k\})p_k - \sum_{j \in S} p_j \hat{x}_j + p_k \hat{x}_k \\ &= \Gamma(S) + p_k(-\frac{1}{2}p_k - p(S \setminus \{k\}) + \hat{x}_k) \end{aligned}$$

- ▶ Thus, if S is a maximizer and $k \in S$, this implies $-\frac{1}{2}p_k - p(S \setminus \{k\}) + \hat{x}_k \leq 0$, which is equivalent to $\hat{x}_k \leq p(S) - \frac{1}{2}p_k$.
- ▶ Hence, S is a maximizer if and only if $\hat{x}_k \leq p(S) \iff k \in S$ holds for all $k \in J$.
- ▶ This shows that $\hat{x}_j \leq \hat{x}_k$ and $k \in S$ imply $j \in S$.
- ▶ Assuming that the jobs are $J = \{1, 2, \dots, n\}$ with $\hat{x}_1 \leq \hat{x}_2 \leq \dots \leq \hat{x}_n$, we only have to test violation for sets $S_k := \{1, 2, \dots, k\}$ for $k = 1, 2, \dots, n-1$. ■

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Extended Formulation

Variables:

- ▶ $x_j \in \mathbb{R}$ for $j \in J$ denotes the half time for job j .
- ▶ $y_{i,j} \in \{0, 1\}$ for $i, j \in J$, $i \neq j$, indicates whether job i is scheduled before job j ($y_{i,j} = 1$) or not ($y_{i,j} = 0$).

Constraints:

$$\frac{1}{2}p_j + \sum_{i \in J \setminus \{j\}} p_i y_{i,j} = x_j \quad \forall j \in J \quad (3a)$$

$$y_{i,j} + y_{j,i} = 1 \quad \forall i, j \in J, i \neq j \quad (3b)$$

$$y_{i,j} \geq 0 \quad \forall i, j \in J, i \neq j \quad (3c)$$

Lemma – Isomorphism to cube

The feasible region of (3) is affinely isomorphic to a cube of dimension $\binom{n}{2}$.
In particular, all vertices have binary y -entries.

Proof:

- ▶ Eliminate the x -variables since they are just determined by (3a).
- ▶ Eliminate all $y_{i,j}$ -variables for $i > j$ since they are determined by (3b).
- ▶ What remains is $0 \leq y_{i,j} \leq 1$ for all $i, j \in J$ with $i < j$. ■

Theorem – Extended formulation for half time polytope

[Wolsey '80s]

The half time polytope has an extended formulation of size $\mathcal{O}(n^2)$.

Proof:

- ▶ Let Q denote the polytope defined by (3).
- ▶ Each half time vector has a preimage by setting $y_{i,j} = 1$ if and only if $h_i < h_j$.
- ▶ It remains to show that every vertex (x, y) of Q (which has binary y -entries due to the lemma) satisfies $x \in P_{\text{ht}}$.
- ▶ Let $D = (J, A)$ be the digraph on node set J with $(i, j) \in A \iff y_{i,j} = 1$.
- ▶ If D contains no directed cycle, then it corresponds to a schedule, and x is its half time vector.
- ▶ Assume that D contains a directed cycle. Since for every $\{i, j\}$, either $(i, j) \in A$ or $(j, i) \in A$ holds, D must even contain a directed cycle of length 3.

Extended Formulation

Reminder: $x_j = \frac{1}{2}p_j + \sum_{i \in J \setminus \{j\}} p_i y_{i,j}$ (3a)

Proof (continued):

- ▶ Consider the cycle $C = \{(i, j), (j, k), (k, i)\} \subseteq A$.
- ▶ If we replace the arc (i, j) by (j, i) , we obtain the new vector $x^{i,j} \in \mathbb{R}^J$ with

$$x_\ell^{i,j} - x_\ell = \begin{cases} p_j & \text{if } \ell = i, \\ -p_i & \text{if } \ell = j, \\ 0 & \text{otherwise,} \end{cases}$$

and similar formulas hold for $x^{j,k}$ (replacing (j, k) by (k, j)) and $x^{k,i}$ (replacing (k, i) by (i, k)).

- ▶ We obtain

$$(p_k(x_\ell^{i,j} - x_\ell) + p_i(x_\ell^{j,k} - x_\ell) + p_j(x_\ell^{k,i} - x_\ell)) / (p_i + p_j + p_k) = 0 \quad \ell = 1, 2, \dots, n$$

and thus

$$\frac{p_k}{p_i + p_j + p_k} x_\ell^{i,j} + \frac{p_i}{p_i + p_j + p_k} x_\ell^{j,k} + \frac{p_j}{p_i + p_j + p_k} x_\ell^{k,i} = x_\ell,$$

which shows that x is a convex combination of other feasible solutions, and thus not a vertex of the projection. ■

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Time Discretization

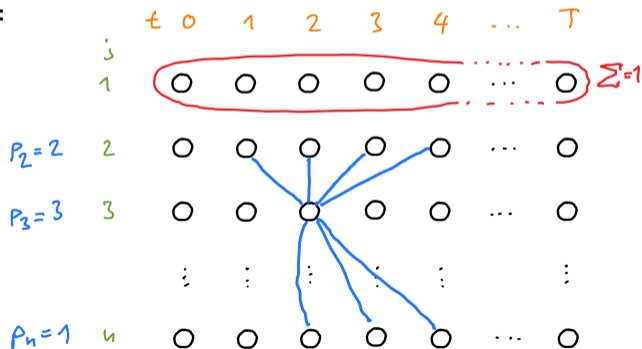
Idea:

- ▶ To avoid continuous variables, we try to model the problem with only binary variables.
- ▶ We **discretize** the allowed starting times, e.g., $s_j \in \mathcal{T} := \{0, 1, 2, \dots, T\}$.
- ▶ In general this is an approximation, but not here due to $p_j \in \mathbb{Z}$.

Variables:

- ▶ $y_{j,t} \in \{0, 1\}$ for $(j, t) \in J \times \mathcal{T}$: $y_{j,t} = 1 \iff$ job j starts **at** time t .

Visualization:



Time Discretization Formulation

Variables:

- ▶ $y_{j,t} \in \{0, 1\}$ for $(j, t) \in J \times \mathcal{T}$: $y_{j,t} = 1 \iff$ job j starts at time t .

IP:

$$\min \sum_{j \in J} \sum_{t \in \mathcal{T}} w_j (t + \frac{1}{2} p_j) y_{j,t} \quad (4a)$$

$$\text{s.t.} \quad \sum_{t \in \mathcal{T}} y_{j,t} = 1 \quad \forall j \in J \quad (4b)$$

$$y_{j,t} + y_{j',t'} \leq 1 \quad \forall (j, t), (j', t') \in J \times \mathcal{T} : j \neq j', t' \in (t - p_{j'}, t + p_j) \quad (4c)$$

$$y_{j,t} \in \{0, 1\} \quad \forall (j, t) \in J \times \mathcal{T} \quad (4d)$$

Proposition – Correctness of time-discretized formulation

Formulation (4) correctly models the minimum weight half time scheduling problem.

Proof:

- ▶ (4a) is correct: $y_{j,t} = 1$ means that j starts at t and thus is half done at $t + \frac{1}{2} p_j$.
- ▶ (4b) ensures that each job is started exactly once.
- ▶ (4c) ensures that jobs j and j' do not overlap. ■